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Linear Projectors in the max-plus Algebra

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Abstract

In general semimodules, we say that the image of a linear operator B and the kernel of a linear operator C are direct factors if every equivalence class modulo C crosses the image of B at a unique point. For linear maps represented by matrices over certain idempotent semifields such as the $(\max,+)$-semiring, we give necessary and sufficient conditions for an image and a kernel to be direct factors. We characterize the semimodules that admit a direct factor (or equivalently, the semimodules that are the image of a linear projector): their matrices have a g-inverse. We give simple effective tests for all these properties, in terms of matrix residuation.

1 Introduction

Classical linear control theory is built on a firm algebraic ground: vector spaces, and modules. There is some evidence that the construction of a 'geometric approach' of $(\max,+)$-linear discrete event systems, in the spirit of Wonham [14], requires the analogue of module theory, for semimodules over idempotent semirings, such as the '$(\max,+)$ semiring' $\mathbb{R}_{\max} = (\mathbb{R}\cup\{-\infty\},\max,+)$. By comparison with modules, the theory of semimodules over idempotent semirings is an essentially fresh subject, in which even the most basic questions are yet unsolved.

Clearly, the *image* of a linear map $F : \mathcal{X} \to \mathcal{Y}$ should be defined as usual: $\operatorname{im} F = \{F(x) \mid x \in \mathcal{X}\}$. But what is the *kernel* of F? Some authors [7, 12] define $\ker F = \{x \in \mathcal{X} \mid F(x) = \varepsilon\}$, where ε is the zero element of $\mathcal{Y}$. This notion is essentially non pertinent for $\mathbb{R}_{\max}$-linear maps, since $\ker F$ is in general trivial, even for 'strongly' non injective maps.

Consider now the following alternative definition

$$\ker F = \left\{(x,x') \in \mathcal{X}^2 \;\middle|\; F(x) = F(x')\right\} . \tag{1}$$

Clearly, $\ker F$ is a semimodule congruence, and we can define the quotient semimodule $\mathcal{X}/\ker F$. Now, trivially, the *canonical isomorphism theorem* holds: $\operatorname{im} F \simeq \mathcal{X}/\ker F$. Thus, in general semimodules, (1) seems to be the appropriate definition. For control applications, it is indeed appropriate, since F typically represents an observation map, by which we wish to quotient some state space.

In this paper, we consider the linear projection problem. Consider three semimodules $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ and two linear maps B, C:

$$\mathcal{U} \xrightarrow{B} \mathcal{X} \xrightarrow{C} \mathcal{Y} . \tag{2}$$

We say that $\operatorname{im} B$ and $\ker C$ are *direct factors* if for all $x \in \mathcal{X}$, there is a unique $\xi \in \operatorname{im} B$ such that $Cx = C\xi$. When it is the case: 1. the map $\Pi_B^C : \mathcal{X} \to \mathcal{X}, x \mapsto z$, which is linear, satisfies $(\Pi_B^C)^2 = \Pi_B^C$, $\Pi_B^C B = B$, $C\Pi_B^C = C$ (Π_B^C is the *projector onto* $\operatorname{im} B$, *parallel to* $\ker C$); 2. we have the isomorphism $\mathcal{X}/\ker C \simeq \operatorname{im} B$; in particular, if $\mathcal{U}$ and $\mathcal{X}$ are free finitely generated semimodules, then the linear map B, which can be identified with a matrix, yields a parametrization of the 'abstract' object $\mathcal{X}/\ker C$.

In [5], a first answer to the projection problem was given, in a nonlinear setting. If $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ are complete lattices, and if B, C are (possibly *nonlinear*) *residuated maps* (see § 2.2 below), it was shown that $\operatorname{im} B$ and $\ker C$ are direct factors iff, setting $\Pi = B\circ(C\circ B)^\sharp\circ C$, (where $F^\sharp$ denotes the residuated map of a map F), we have $C\circ\Pi = C$ and $\Pi\circ B = B$. Even in the simplest case of $\mathbb{R}_{\max}$-linear operators over free finitely generated $\mathbb{R}_{\max}$-semimodules (that is, when B and C are matrices with entries in $\mathbb{R}_{\max}$), the operator $\Pi = B\circ(C\circ B)^\sharp\circ C$ is a complicated object (a min-max function in the sense of Olsder and Gunawardena — see e.g. [10]), and the test $C\circ\Pi = C$, $\Pi\circ B = B$, is computationally difficult (an example of non trivial direct factors was given in [5], for $\mathcal{U} = \mathcal{Y} = (\mathbb{R}_{\max})^2$, $\mathcal{X} = (\mathbb{R}_{\max})^3$, the proof that $\operatorname{im} B$ and $\ker C$ are direct factors involved a tedious computation of equivalence classes, together with a geometrical argument).

In this paper, we give a much simpler test for matrices: $\operatorname{im} B$ and $\ker C$ are direct factors iff there exist two matrices L, K such that $B = LCB$ and $C = CBK$ (we denote with the same symbol B the matrix B and the linear map $x \mapsto Bx$). Then, $\Pi_B^C = LC = BK$. The existence of the matrices K, L can be checked very simply (in polynomial time) using residuation of *matrices* (and not of linear maps).

As a by-product, we solve the following problem, which was left open in [5]: *given a matrix B, does there exist a projector onto* $\operatorname{im} B$?; or, equivalently *does there exist a matrix C such that* $\operatorname{im} B$ *and* $\ker C$ *are direct factors?* The answer is positive iff B admits a g-inverse, that is, iff $B = BXB$, for some matrix X. The existence of a g-inverse can also be checked (simply) in polynomial time using matrix residuation.

The proofs are critically based on a *linear extension* theorem, which states that a linear form F on a finitely generated subsemimodule of $(\mathbb{R}_{\max})^n$ can be represented by a row vector G: $F(x) = Gx$. This result was proved by Kim [8, Lemma 1.3.2] for matrices with entries in the Boolean semiring. Cao, Kim and Roush [4, Th. 4.7.4] proved a variant of this result for the semiring $([0,1], \max, \times)$. As in the case of [8, 4], the proof consists in proving that the maximal linear subextension is an extension. This seems to require very strong properties on the dioid (lattice distributivity, invertibility of product).

Note that certain results pertaining to kernels rely upon a linear extension theorem whereas this theorem is not required to prove dual results on images

In §2, we introduce the algebraic notions used in the paper. In §3, we prove the linear extension theorem, and derive factorization theorems for linear maps. In §4, we characterize direct factors. In §5, we relate the existence of projectors to the existence of g-inverses.

2 Algebraic Preliminaries

We briefly and informally recall the few algebraic results needed here. More details can be found in [1] for dioids and ordered sets, and in [7] for semirings and semimodules. A seminal reference in residuation theory is [3]. See also [6].

A *semiring* is a set $\mathcal{S}$ equipped with two laws $\oplus$, $\otimes$, such that: $(\mathcal{S}, \oplus)$ is a commutative monoid (the zero is denoted ε); $(\mathcal{S}, \otimes)$ is a (possibly noncommutative) monoid (the unit is denoted e); $\otimes$ is right and left distributive over $\oplus$; and the zero is absorbing. A semiring in which non zero elements have an inverse is a *semifield*. A semiring $\mathcal{S}$ is *idempotent* if $\forall a \in \mathcal{S}, a \oplus a = a$. Idempotent semirings are also called *dioids*. In this paper, we will mostly consider dioids such as $\mathbb{R}_{\max}$, which is also a semifield.

2.1 Order properties of dioids

A dioid (or more generally, an idempotent additive monoid) is equipped with the *natural* order relation:

$$a \le b \iff a \oplus b = b \ . \tag{3}$$

Then, $a \oplus b$ coincides with the upper bound $a \vee b$ for the natural order $\le$. Note that ε is the bottom element of $\mathcal{D}$: $\forall x \in \mathcal{D},\ \varepsilon \le x$.

Moreover, if $\mathcal{D}$ is a semifield, $(\mathcal{D}, \le)$ is a lattice. Indeed, for all non zero a, b, $a \wedge b = (a^{-1} \vee b^{-1})^{-1} = (a^{-1} \oplus b^{-1})^{-1}$; if a or b is zero, $a \wedge b = \varepsilon$. This shows that $(\mathcal{D}, \le)$ is a lattice. We say that the idempotent semifield $\mathcal{D}$ is distributive if the lattice $(\mathcal{D}, \le)$ is distributive [1].

For the sake of symmetry, we will complete an idempotent semifield $\mathcal{D}$ with a maximal element $\top$ (for "top"), which satisfies $a \oplus \top = \top, \forall a \in \mathcal{D} \cup \{\top\}$, and $a \otimes \top = \top \otimes a = \top, \forall a \in (\mathcal{D} \cup \{\top\}) \setminus \{\varepsilon\}$. We denote $\overline{\mathcal{D}} = \mathcal{D} \cup \{\top\}$ this dioid, and we will call it the *top completion* of $\mathcal{D}$.

In $\overline{\mathcal{D}}$, the product also distributes with respect to $\wedge$:

$$\forall a, b, c \in \overline{\mathcal{D}}, \qquad \begin{aligned} a(c \wedge d) &= ac \wedge ad \ , \\ (c \wedge d)a &= ca \wedge da \end{aligned} \tag{4}$$

(this property does not hold in general dioids).

2.2 Residuation

Definition 1. We say that a dioid $\mathcal{D}$ is *residuated* if

1. for all a and b in $\mathcal{D}$, $\{x \in \mathcal{D} \mid ax \le b\}$ admits a maximal element denoted $a \backslash b$;
2. $\{x \in \mathcal{D} \mid xa \le b\}$ admits a maximal element denoted b/a;
3. $(\mathcal{D}, \le)$ is a lattice.

Then, the maximal element of $\{x \in \mathcal{D} \mid axc \le b\}$ exists and can be denoted $a \backslash b / c$ which can be read indifferently as $(a \backslash b)/c$ or $a \backslash (b/c)$.

The top completion $\overline{\mathcal{S}}$ of an idempotent semifield $\mathcal{S}$ is residuated, with $a \backslash x = a^{-1}x$ if a is invertible, $\varepsilon \backslash x = \top$, and $\top \backslash x = \varepsilon$ if $x \ne \top$, $\top \backslash \top = \top$ (similar formulæ for $/$).

Consider the following linear equations in X:

$$AX = B \ , \tag{5a}$$

$$XC = D \ , \tag{5b}$$

$$AXC = F \ , \tag{5c}$$

where $X, A, \ldots$ are (possibly rectangular) matrices with entries in a residuated dioid $\mathcal{D}$.

We extend the $\cdot\backslash\cdot$ and $\cdot/\cdot$ notation to matrices:

$$A \backslash B \stackrel{\text{def}}{=} \bigvee \{X \mid AX \le B\} \ , \tag{6a}$$

$$D/C \stackrel{\text{def}}{=} \bigvee \{X \mid XC \le D\} \ , \tag{6b}$$

$$A \backslash F / C \stackrel{\text{def}}{=} \bigvee \{X \mid AXC \le F\} \ . \tag{6c}$$

Explicitly, we have the following formulæ, which relate the residuation of matrices to the residuation of scalars:

$$(A \backslash B)_{ij} = \bigwedge_k A_{ki} \backslash B_{kj} \ , \tag{7a}$$

$$(D/C)_{ij} = \bigwedge_l D_{il} / C_{jl} \ , \tag{7b}$$

$$(A \backslash F / C)_{ij} = \bigwedge_{kl} A_{ki} \backslash F_{kl} / C_{jl} \ . \tag{7c}$$

To decide whether the matrix equations (5) have a solution, it suffices to check that the maximal subsolution satisfies the equality.

Proposition 2. *Take five matrices A, B, C, D, F as above, with entries in a residuated dioid. Then:*

$$\exists X,\ AX = B \iff A(A\backslash B) = B\ , \tag{8a}$$
$$\exists X,\ XC = D \iff (D/C)C = D\ , \tag{8b}$$
$$\exists X,\ AXC = F \iff A(A\backslash F/C)C = F\ . \tag{8c}$$

2.3 Semimodules

In this section, $\mathcal{S}$ denotes an arbitrary semiring. A *right $\mathcal{S}$-semimodule*, or a *right semimodule over $\mathcal{S}$*, is a commutative monoid $(\mathcal{E}, \oplus)$, together with an external law $\mathcal{E}\times\mathcal{S} \to \mathcal{E}, (u, s) \mapsto u.s$, which satisfies, for all $u, v \in \mathcal{E}$, $s, t \in \mathcal{S}$, $u.(st) = (u.s).t$, $(u \oplus v).s = u.s \oplus v.s$, $u.(s \oplus t) = u.s \oplus u.t$, $\varepsilon.s = \varepsilon$, $u.\varepsilon = \varepsilon$, $u.e = u$.

Left $\mathcal{S}$-semimodules are defined dually. For simplicity, we will simply speak of *semimodule* when the underlying semiring $\mathcal{S}$ and the side (right vs. left) are clear from the context.

In a semimodule over a dioid, addition is idempotent. Indeed, $a \oplus a = a.e \oplus a.e = a.(e \oplus e) = a.e = a$.

A map F from a (right $\mathcal{S}$-) semimodule $\mathcal{E}$ to a (right $\mathcal{S}$-) semimodule $\mathcal{F}$ is *linear* if it is additive ($\forall u, v \in \mathcal{E},\quad F(u \oplus v) = F(u) \oplus F(v)$) and right-homogeneous ($\forall u \in \mathcal{E}, s \in \mathcal{S}, F(u.s) = F(u).s$). The set of linear maps $\mathcal{E} \to \mathcal{F}$ is denoted $\mathrm{Hom}\,(\mathcal{E}, \mathcal{F})$.

A *generating family* of a semimodule $\mathcal{E}$ is a family $\{u_i\}_{i\in I}$ of elements of $\mathcal{E}$ such that each element $v \in \mathcal{E}$ writes as a finite linear combination $v = \bigoplus_{i\in I} u_i.s_i$, with $s_i \in \mathcal{S}$ ('finite' means that $\{i \in I \mid s_i \neq \varepsilon\}$ is finite, even if I is infinite). A generating family $\{u_i\}_{i\in I}$ is a *basis* if $\bigoplus_{i\in I} u_i.s_i = \bigoplus_{i\in I} u_i.t_i$, with $\{i \in I \mid s_i \neq \varepsilon\}$ and $\{i \in I \mid t_i \neq \varepsilon\}$ finite, implies $s_i = t_i$, for all $i \in I$. A semimodule is *finitely generated* (*f.g.*, for short) if it has a finite generating family. A semimodule is *free* if it has a basis.

The term *free* for $\mathcal{E}$ arises from the following universal property: given an arbitrary family $\{g_i\}_{i\in I}$ of elements of a semimodule $\mathcal{F}$, there is a unique linear map $F : \mathcal{E} \to \mathcal{F}$ such that $F(u_i) = g_i, \forall i \in I$.

All semimodules with a basis of n elements are isomorphic to $\mathcal{S}^n$, equipped with the laws: $\forall u, v \in \mathcal{S}^n, s \in \mathcal{S}$, $(u \oplus v)_i = u_i \oplus v_i$, $(u.s)_i = u_i.s$. A linear map $F : \mathcal{S}^p \to \mathcal{S}^n$ writes $F(x) = Ax$, where A is a $n \times p$ matrix with entries in $\mathcal{S}$.

We will use the following notation, for matrices:

transpose:	$(A^T)_{ij} = A_{ji}$,
kernel:	$\ker A = \left\{(x, y) \in (\mathcal{S}^n)^2 \mid Ax = Ay\right\}$,
image:	$\mathrm{im}\, A = \{Ax \mid x \in \mathcal{S}^n\}$.

That is, matrix A is identified with the linear map $x \mapsto Ax$. We will use this convention systematically in the sequel.

3 Linear Extension Theorem and Factorization of Linear Maps

Let us begin with an elementary and apparently innocent property.

Proposition 3. *Let $\mathcal{S}$ denote an arbitrary semiring. Consider a free $\mathcal{S}$-semimodule $\mathcal{F}$, two $\mathcal{S}$-semimodules $\mathcal{G}$, $\mathcal{H}$, and two linear maps $F : \mathcal{F} \to \mathcal{H}$, $G : \mathcal{G} \to \mathcal{H}$. The following assertions are equivalent:*

1. $\mathrm{im}\, F \subset \mathrm{im}\, G$;

2. *there exists a linear map $H : \mathcal{F} \to \mathcal{G}$ such that $F = G \circ H$.*

Proof. Clearly, (2) implies (1). Conversely, taking a basis of $\mathcal{F}$, $\{u_i\}_{i\in I}$, for all $i \in I$, there exists $h_i \in \mathrm{im}\, G$ such that $F(u_i) = G(h_i)$. Since $\mathcal{F}$ is free, setting $H(u_i) = h_i$, for all $i \in I$, we define a linear map $H : \mathcal{F} \to \mathcal{G}$, which satisfies $F = G \circ H$. □

The dual of Prop. 3 requires some conditions on the semiring, as shown by the following counterexample.

Example 4. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ denote three free semimodules, and consider two maps $F : \mathcal{H} \to \mathcal{F}$, $G : \mathcal{H} \to \mathcal{G}$. The inclusion $\ker G \subset \ker F$ need not imply the existence of a linear map $H : \mathcal{G} \to \mathcal{F}$ such that $F = H \circ G$. Consider the semiring $\mathbb{N}_{\max} = (\mathbb{N} \cup \{-\infty\}, \max, +)$ equipped with the laws $\oplus = \max$, $\otimes = +$, $\mathcal{F} = \mathcal{G} = \mathcal{H} = \mathbb{N}_{\max}$, $G(x) = x + 2$, $F(x) = x + 1$. We have $\ker G \subset \ker F$ (in fact, $\ker G = \ker F = \{(x, x) \mid x \in \mathbb{N}_{\max}\}$) but there exists no linear map H such that $F = H \circ G$. Indeed, any linear map $H : \mathbb{N}_{\max} \to \mathbb{N}_{\max}$ writes $H(x) = x + a$ where $a = H(0) \in \mathbb{N}_{\max}$. We obtain $F(0) = 1 = 2 + a$: a contradiction.

We will derive the dual of Prop. 3 from the following semimodule version of the Hahn-Banach theorem.

Theorem 5 (Linear Extension). *Let $\mathcal{S}$ be a distributive idempotent semifield. Let $\mathcal{F}, \mathcal{G}$ denote two free f.g. $\mathcal{S}$-semimodules, and let $\mathcal{H} \subset \mathcal{G}$ be a f.g. subsemimodule. For all $F \in \mathrm{Hom}\,(\mathcal{H}, \mathcal{F})$, there exists $G \in \mathrm{Hom}\,(\mathcal{G}, \mathcal{F})$ such that $\forall x \in \mathcal{H},\ G(x) = F(x)$.*

Proof. It suffices to prove the result when $\mathcal{G} = \mathcal{S}^n$ and $\mathcal{F} = \mathcal{S}$. Since $\mathcal{H} \subset \mathcal{G}$ is f.g., we have $\mathcal{H} = \mathrm{im}\, H$, for some $H \in \mathcal{S}^{n\times p}$. Clearly, we can assume that H has no zero row. In this case, for all $1 \leq i \leq n$,

$$L(i) = \left\{ j \;\middle|\; 1 \leq j \leq p,\ H_{ij} \neq \varepsilon \right\} \neq \emptyset\ .$$

Let $H_{\cdot j}$ denote the j-th column of H, and let $F(H)$ denote the row-vector whose j-th entry is $F(H_{\cdot j})$. We have to prove the existence of a row-vector $G \in \mathcal{S}^{1\times n}$ such that

$$F(H) = GH\ . \tag{9}$$

Using (8b), this is equivalent to

$$F(H) = \big(F(H)/H\big)H \; . \tag{10}$$

Using (7b), we get:

$$\Big(\big(F(H)/H\big)H\Big)_j = \bigoplus_k \left(\bigwedge_{l\in L(k)} F(H)_l H_{kl}^{-1}\right) H_{kj} \; .$$

Using the distributivity of product with respect to $\wedge$ (see (4)), we obtain:

$$\Big(\big(F(H)/H\big)H\Big)_j = \bigoplus_k \bigwedge_{l\in L(k)} F(H)_l H_{kl}^{-1} H_{kj} \; . \tag{11}$$

Let Φ denote the set of maps $\varphi : \{1,\dots,n\} \to \cup_k L(k)$, such that $\varphi(k) \in L(k)$, for all k. Since the lattice $(\mathcal{S}, \leq)$ is distributive, we have:

$$\begin{aligned}
\Big(\big(F(H)/H\big)H\Big)_j &= \bigwedge_{\varphi\in\Phi} \bigoplus_k F(H)_{\varphi(k)} H_{k\varphi(k)}^{-1} H_{kj} \\
&= \bigwedge_{\varphi\in\Phi} \bigoplus_k F(H_{\cdot\varphi(k)}) H_{k\varphi(k)}^{-1} H_{kj} \\
&= \bigwedge_{\varphi\in\Phi} F\left(\bigoplus_k H_{\cdot\varphi(k)} H_{k\varphi(k)}^{-1} H_{kj}\right)
\end{aligned}$$

(the last equality is by linearity of F on $\operatorname{im} H$). To show that $(F(H)/H)H \geq F(H)$ (the other inequality is trivial), it remains to check that for all φ and j,

$$\bigoplus_k H_{\cdot\varphi(k)} H_{k\varphi(k)}^{-1} H_{kj} \geq H_{\cdot j} \; . \tag{12}$$

If $H_{ij} = \varepsilon$, then the inequality (12) is plain. If $H_{ij} \neq \varepsilon$, we choose $k = i$, and obtain $H_{i\varphi(i)} H_{i\varphi(i)}^{-1} H_{ij} = H_{ij}$, which shows that (12) holds and the proof is complete. □

Corollary 6. *Let $\mathcal{S}$ be an idempotent distributive semifield. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ denote three free f.g. $\mathcal{S}$-semimodules, and consider two maps $F : \mathcal{H} \to \mathcal{F}$, $G : \mathcal{H} \to \mathcal{G}$. The following assertions are equivalent:*

1. $\ker G \subset \ker F$*;*

2. *there exists a linear map $H : \mathcal{G} \to \mathcal{F}$ such that $F = H \circ G$.*

Proof. Clearly, 2 implies 1. Conversely, assume that $\ker G \subset \ker F$. Then, there exists a map $K \in \operatorname{Hom}(\operatorname{im} G, \mathcal{F})$ such that $K(G(x)) = F(x)$, for all $x \in \mathcal{H}$. Indeed, for any $y = G(x) \in \operatorname{im} G$, define $K(y) = F(x)$. Since $\ker G \subset \ker F$, the value $K(y)$ is independent of the choice of x such that $y = F(x)$. Clearly, the map K is linear. By Theorem 5, K admits a linear extension $H \in \operatorname{Hom}(\mathcal{G}, \mathcal{F})$. For all $x \in \mathcal{H}$, we have $H \circ G(x) = K(G(x)) = F(x)$, hence $H \circ G = F$. □

4 Direct Factors in Semimodules and Linear Projectors

Definition 7. Let X be a subset in a semimodule $\mathcal{X}$ and $\mathcal{E}$ be an equivalence relation in $\mathcal{X}$. We say that x in $\mathcal{X}$ has a *projection on X parallel to* $\mathcal{E}$ if there exists ξ in X such that $\xi \; \mathcal{E} \; x$. We say that X *crosses* $\mathcal{E}$ if there exists such a projection for all x in $\mathcal{X}$. We say that X is *transverse to* $\mathcal{E}$ if the projection of any x is unique whenever it exists. Finally, we say that X and $\mathcal{E}$ are *direct* factors if existence and uniqueness of the projection is ensured for all x in $\mathcal{X}$.

In the previous definition, consider $X = \operatorname{im} B$ for $B \in \operatorname{Hom}(\mathcal{U}, \mathcal{X})$ and $\mathcal{E} = \ker C$ for $C \in \operatorname{Hom}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are semimodules over a semiring $\mathcal{S}$. If $\operatorname{im} B$ and $\ker C$ are direct factors, Π_B^C denotes the corresponding projector. It is straightforward to check that $\Pi_B^C \in \operatorname{Hom}(\mathcal{X}, \mathcal{X})$. Also, from the very definition, it comes that

$$B = \Pi_B^C B \; ; \quad C = C\Pi_B^C \; . \tag{13}$$

However, the existence of a linear projector Π (that is, such that $\Pi^2 = \Pi$) satisfying (13) is not a sufficient condition for $\operatorname{im} B$ and $\ker C$ to be direct factors. Indeed, for *any* B and C, the identity over $\mathcal{X}$ satisfies (13).

Theorem 8 (Existence). *Let $\mathcal{S}$ denote an arbitrary semiring. Let $B \in \operatorname{Hom}(\mathcal{U}, \mathcal{X})$ and $C \in \operatorname{Hom}(\mathcal{X}, \mathcal{Y})$ where $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ are free f.g. $\mathcal{S}$-semimodules. The following statements are equivalent:*

1. *there exists $K \in \operatorname{Hom}(\mathcal{X}, \mathcal{U})$ such that*
$$C = CBK \; ; \tag{14}$$

2. $\operatorname{im} C = \operatorname{im} CB$*;*

3. $\operatorname{im} B$ *crosses* $\ker C$.

Moreover, if $\mathcal{S}$ is a residuated dioid, a practical test for checking that the above conditions hold true is by trying the equality

$$C = CB\big((CB)\backslash C\big) \; . \tag{15}$$

Proof.

$1 \Rightarrow 2$ The assumption implies that $\operatorname{im} C \subset \operatorname{im} CB$ but the converse inclusion is trivial. Hence equality holds true.

$2 \Rightarrow 3$ From the assumption, it follows that for all $x \in \mathcal{X}$, there exists $u \in \mathcal{U}$ such that $Cx = CBu$. The projection of x we are looking for is $\xi = Bu$.

$3 \Rightarrow 1$ By assumption, for all $x \in \mathcal{X}$, there exists $u \in \mathcal{U}$ such that $Cx = CBu$. That is to say, $\operatorname{im} C \subset \operatorname{im} CB$. From Prop. 3, it follows that there exists $K \in \operatorname{Hom}(\mathcal{X}, \mathcal{U})$ such that $C = CBK$.

The practical test follows from statement 1, together with (8a). $\square$

Theorem 9 (Uniqueness). *The mappings B and C are as in the previous theorem. But now $\mathcal{S}$ is an idempotent distributive semifield. The following statements are equivalent:*

1. $\operatorname{im} B$ *is transverse to* $\ker C$*;*
2. $\ker B = \ker CB$*;*
3. *there exists* $L \in \operatorname{Hom}(\mathcal{Y}, \mathcal{X})$ *such that*

$$B = LCB \ . \tag{16}$$

A practical test for checking that they hold true is by trying the equality

$$B = \big(B/(CB)\big)CB \ . \tag{17}$$

Proof.

$1 \Rightarrow 2$ By assumption, if there exist u and u' such that $CBu = CBu'$, since $x = Bu$ has a unique projection on $\operatorname{im} B$ parallel to $\ker C$, it should be that $Bu = Bu'$. This means that $\ker CB \subset \ker B$. But the converse inclusion is trivial, hence equality holds true.

$2 \Rightarrow 3$ If $\ker B = \ker CB$, from Cor. 6, there exists $L \in \operatorname{Hom}(\mathcal{Y}, \mathcal{X})$ such that $B = LCB$.

$3 \Rightarrow 1$ For some x, suppose there exist two projections Bu and Bu' on $\operatorname{im} B$ parallel to $\ker C$. Then $CBu = CBu'$, hence $LCBu = LCBu'$, thus $Bu = Bu'$ and the projection is unique.

The practical test follows from statement 1, together with (8b). $\square$

Corollary 10. *If* $\operatorname{im} B$ *and* $\ker C$ *are direct factors, then*

$$\Pi_B^C = LC = BK = \big(B/(CB)\big)C = B\big((CB)\backslash C\big) \ . \tag{18}$$

Proof. If B and C are direct factors, then (14) and (16) both hold true. Consider $\Pi = BK$. From (16), $\Pi = LCBK$, and then from (14), $\Pi = LC$. Thus, $\Pi = BK = LC = \Pi^2$. In addition, this shows that, for any x, Πx belongs to $\operatorname{im} B$ and, moreover, (14) shows that $C\Pi x = Cx$, which means that the projection on $\operatorname{im} B$ is parallel to $\ker C$. Thus this projector Π is indeed Π_B^C. The last two expressions in (18) result from the choice of maximal L and K previously mentioned. $\square$

Remark 11. Gathering the results in (15) on the one hand, of (13) and (18) on the other hand, one has that

$$C = CB\big((CB)\backslash C\big) = C\big(B/(CB)\big)C \ . \tag{19a}$$

Similarly, with (17) on the one hand, (13) and (18) again on the other hand, one obtains

$$B = \big(B/(CB)\big)CB = B\big((CB)\backslash C\big)B \ . \tag{19b}$$

However, while the pair of leftmost equations have been proved to be a test that $\operatorname{im} B$ and $\ker C$ are direct factors, there is no evidence at this moment that the other pairs of equations can play the same role.

Remark 12 (Duality). If $\mathcal{S}$ is an idempotent distributive semifield, by transposition, it is straightforward to check that $\operatorname{im} B$ crosses $\ker C$ if, and only if, $\operatorname{im} C^T$ is transverse to $\ker B^T$. Likewise, $\operatorname{im} B$ is transverse to $\ker C$ if, and only if, $\operatorname{im} C^T$ crosses $\ker B^T$. Finally, $\operatorname{im} B$ and $\ker C$ are direct factors if, and only if, $\operatorname{im} C^T$ and $\ker B^T$ are so. In this case, $\big(\Pi_B^C\big)^T = \Pi_{C^T}^{B^T}$ (in general, $(M/N)^T = N^T\backslash M^T$).

5 Direct Factors and g-Inverses

In this section, we answer the question of when a semimodule $\operatorname{im} B$ admits a direct factor $\ker C$. Unlike in the case of usual linear spaces, this question cannot receive a positive answer for any linear operator B. An explicit test is given to characterize homomorphisms such that their images admit a direct factor.

Definition 13. Let $\mathcal{U}$, $\mathcal{X}$ denote two semimodules over an arbitrary semiring $\mathcal{S}$. Let $B \in \operatorname{Hom}(\mathcal{U}, \mathcal{X})$.

1. An element $F \in \operatorname{Hom}(\mathcal{X}, \mathcal{U})$ such that $BFB = B$ is called a *g-inverse* of B;
2. when B admits a g-inverse, it is called *regular*;
3. a g-inverse F which satisfies $FBF = F$ is called a *reflexive g-inverse*;
4. when $\mathcal{S}$ is a dioid[1], an element $F \in \operatorname{Hom}(\mathcal{X}, \mathcal{U})$ such that $BFB \leq B$ is called a *g-subinverse* of B.

Theorem 14. *Let $\mathcal{U}$, $\mathcal{X}$ denote free f.g. semimodules over a residuated dioid $\mathcal{D}$. Then:*

1. *Any* $B \in \operatorname{Hom}(\mathcal{U}, \mathcal{X})$ *admits a maximal g-subinverse. It is denoted* B^{g}*. In matrix terms:* $B^{\mathrm{g}} = B\backslash B/B$.
2. *When B is regular, B^{g} is the greatest g-inverse and B^{r}, defined by $B^{\mathrm{g}}BB^{\mathrm{g}}$, is the greatest reflexive g-inverse of B.*

Proof.

1. This is an immediate consequence of (8c).
2. If B^{g} is a g-inverse, then B^{r} is the maximal reflexive g-inverse: indeed, B^{r} is a reflexive g-inverse since

$$\begin{aligned} BB^{\mathrm{r}}B &= (BB^{\mathrm{g}}B)B^{\mathrm{g}}B = BB^{\mathrm{g}}B = B \ , \\ B^{\mathrm{r}}BB^{\mathrm{r}} &= B^{\mathrm{g}}(BB^{\mathrm{g}}B)B^{\mathrm{g}}BB^{\mathrm{g}} = B^{\mathrm{g}}(BB^{\mathrm{g}}B)B^{\mathrm{g}} \\ &= B^{\mathrm{g}}BB^{\mathrm{g}} = B^{\mathrm{r}} \ . \end{aligned}$$

[1] In this case, the monoid $(\operatorname{Hom}(\mathcal{U}, \mathcal{X}), \oplus)$ is naturally ordered by $F \leq G$ iff $F \oplus G = G$.

It is maximal since, if F is another reflexive g-inverse, then $F \leq B^{\mathrm{g}}$ because B^{g} is the maximal g-inverse. It follows that

$$F = FBF \leq B^{\mathrm{g}} B B^{\mathrm{g}} = B^{\mathrm{r}} .$$

Finally, if B is regular, then $BB^{\mathrm{r}}B = B$ and $B^{\mathrm{r}}BB^{\mathrm{r}} = B^{\mathrm{r}}$. □

Theorem 15. *Let $\mathcal{S}$ be a distributive semifield, $\mathcal{U}$, $\mathcal{X}$ be free f.g. $\mathcal{S}$-semimodules, and $B \in \operatorname{Hom}(\mathcal{U}, \mathcal{X})$. This B is regular iff* $\operatorname{im} B$ *admits a direct factor* $\ker C$*, where $C \in$* $\operatorname{Hom}(\mathcal{X}, \mathcal{Y})$ *and $\mathcal{Y}$ is a free f.g. $\mathcal{S}$-semimodule. One can take for C any reflexive g-inverse of B.*

Proof. Let B^{ρ} be any reflexive g-inverse of B. Then the statements of Theorems 8 and 9 are satisfied with $\mathcal{Y} = \mathcal{U}$, $C = B^{\rho}$, $L = B$, $K = B^{\rho}$, and $\ker B^{\rho}$ is the direct factor we are looking for.

Conversely, if $\operatorname{im} B$ admits a direct factor $\ker C$, by (16), $B = LCB$, and then by (18), $B = BKB$, which shows that B is regular. □

Remark 16. When B is put in the form

$$P \begin{pmatrix} D & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} Q ,$$

where P and Q are permutation matrices and D has no zero rows and columns, it is easy to check that D^{g} — and a fortiori $D^{\mathrm{r}} \leq D^{\mathrm{g}}$ — have no $\top$ entries and that a particular reflexive g-inverse of B is

$$B' = Q^{-1} \begin{pmatrix} D^{\mathrm{r}} & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} P^{-1} .$$

Remark 17. The maximal reflexive g-inverse does not always coincides with the maximal g-inverse. For example, take $\mathcal{S} = \mathbb{R}_{\max}$, and consider

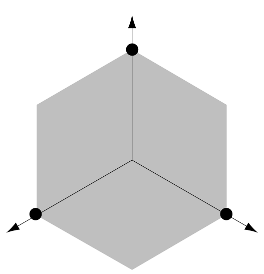
$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; B^{\mathrm{g}} = \begin{pmatrix} -1 & -2 & 0 \\ -2 & -2 & -2 \\ \mathbf{-1} & -2 & -1 \end{pmatrix} \neq B^{\mathrm{r}} = \begin{pmatrix} -1 & -2 & 0 \\ -2 & -2 & -2 \\ \mathbf{-2} & -2 & -1 \end{pmatrix} .$$

Example 18. The following matrix with entries in $\mathbb{R}_{\max}$,

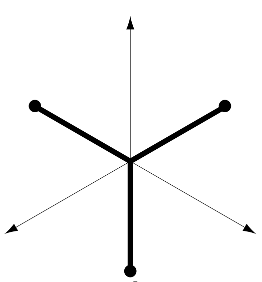
$$B = \begin{pmatrix} 0 & n & 0 \\ 0 & 0 & n \\ n & 0 & 0 \end{pmatrix} ,$$

is regular whenever $n \geq 0$ and not regular otherwise:

$$B^{\mathrm{g}} = \begin{pmatrix} -2n & -2n & -n \\ -n & -2n & -2n \\ -2n & -n & -2n \end{pmatrix} \text{ if } n \geq 0 \text{ and } \begin{pmatrix} n & n & n \\ n & n & n \\ n & n & n \end{pmatrix} \text{ if } n < 0 .$$



regular case



nonregular case

Observe that the image of any homomorphism being invariant by translation along the first diagonal, it is enough to represent $\operatorname{im} B$ by its projection on any plane orthogonal to that diagonal (see figure).

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