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KERNELS, IMAGES AND PROJECTIONS IN DIOIDS

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Abstract

We consider the projection problem for linear spaces and operators over dioids such as the $(\max, +)$ semiring. We give existence and uniqueness conditions for the projection onto the image of an operator, parallel to the kernel of another one, together with an explicit formula for the projector. The theory is not limited to linear operators: the result holds more generally for residuated operators over complete dioids. Illustrative examples are provided.

1 Introduction

One of the most regrettable gaps in the theory of $(\max, +)$ linear Discrete Event Systems is the lack of a geometric understanding, in the spirit of the 'geometric approach' initiated by Wonham [17] for conventional linear systems. The main obstacle in this direction is the lack of a powerful theory on images and kernels of linear operators over the '$(\max, +)$ semiring' $\mathbb{R}_{\max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \max, +)$, similar to the rank theory for vector spaces or to the theory of modules over principal rings.

Finite dimensional images, or equivalently, finitely generated $\mathbb{R}_{\max}$-semimodules, have been seriously investigated. We refer the reader to [14, 15, 16] for existing results (existence of basis, classification tools). Comparatively, kernels seem to have attracted very little attention. Indeed, in the $(\max, +)$ context, one has to define the kernel (or perhaps the 'bikernel') of a linear mapping A as an equivalence relation, namely as the set of pairs (x, y) such that $Ax = Ay$ (the usual definition $\ker A = \{x \mid Ax = \varepsilon\}$, where ε denotes the zero element, carries little information in the $(\max, +)$ case, due the noninvertibility of addition).

In this paper, we study the simplest geometrical problem which consists in projecting onto the image of an operator $B : \mathcal{U} \to \mathcal{X}$ parallel to the kernel of an operator $C : \mathcal{X} \to \mathcal{Y}$. For linear operators over vector spaces, the answer is well known: the existence of such a projector $\Pi : \mathcal{X} \to \operatorname{im} B$, parallel to $\ker C$, is equivalent to the direct-sum decomposition $\mathcal{X} = \operatorname{im} B + \ker C$. Moreover, as soon as B is injective and C is surjective[1], we have

$$\Pi = B(CB)^{-1}C \ . \tag{1}$$

The main difficulty in extending this result is that linear operators over idempotent semimodules are generically not invertible (even not injective, and not surjective). However, they satisfy a suitably relaxed invertibility condition (residuability), which is enough for our purpose. We provide a characterization based on the residuated maps of B, C, and give a formula similar to (1) involving the residual of CB instead of its inverse. The class of residuated mappings (over general ordered structures) is indeed much wider than that of linear 'continuous'[2] operators over idempotent semimodules.

The paper is organized as follows: in §2, we recall the few algebraic elements needed in the paper. Kernels and images are introduced in §3: intrinsic (operator independent) characterizations are provided in terms of factorizations (or Green classes) of residuated mappings. In §4, the operator of projection parallel to the kernel of a residuated operator in isolation (without reference to an image) is determined. Dually, in §5, the operator of projection onto the image of a residuated operator in isolation (without reference to a kernel) is given. These two projection results use the order relation to canonically select a (maximal or minimal) solution to these ill-posed problems, in a way very much analogous to the specification of conventional quasi-inverses by norm minimization arguments. Next (§6), we address the complete projection problem. We conclude with illustrating examples for $\mathbb{R}_{\max}$ semimodules.

Of course, projection operators are instrumental in linear algebra and in control, and the study of their $(\max, +)$ and dioid analogues needs almost no justification. However, we would like to mention a specific application to aggregation and lumpability problems. One may formulate aggregation problems for Markov chains (and more generally, for linear dynamical systems) in a purely algebraic way using projection operators, as in [6]. With the projection theorem given here at hand, this approach

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[1] These conditions are not too restrictive in the case of linear operators over vector spaces. Indeed, what is important is the geometric objects $\operatorname{im} B$ and $\ker C$ rather than the representative operators B and C themselves.

[2] Precisely, lower semicontinuous, as defined below.

extends verbatim to the $(\max, +)$ case. Then, one obtains aggregation conditions for Timed Event Graphs, i.e. conditions for the existence of (usually physically meaningful) aggregated variables (typically the maximal completion time of all the tasks within a certain relevant class), from which the complete behavior can be retraced. This is reported in [4].

2 Algebraic preliminaries

We briefly and informally recall the few algebraic results needed here. More details can be found in [1] for dioids and ordered sets, and in [9] for semirings and semimodules. A seminal reference in residuation theory is [2]. See also [5].

2.1 Order properties of dioids

A *dioid* $(\mathcal{D}, \oplus, \otimes)$ is a semiring in which addition is idempotent: $a \oplus a = a$. The zero and unit elements will be denoted ε and e respectively. A dioid (or more generally, an idempotent additive monoid) is equipped with the *natural* order relation:

$$a \leq b \iff a \oplus b = b \ . \tag{2}$$

Then, $a \oplus b$ coincides with the upper bound $\sup\{a, b\}$ for the natural order $\leq$. Note that ε is the bottom element of $\mathcal{D}$: $\forall x \in \mathcal{D},\ \varepsilon \leq x$. Indeed, (2) sets up a one to one correspondence between ordered sets $(\mathcal{D}, \leq)$ with upper bounds for any pairs of elements and a bottom element for the whole set on the one hand, and commutative idempotent monoids on the other hand. We say that an idempotent monoid is *complete* whenever an arbitrary (possibly infinite) set $X \subset \mathcal{D}$ admits an upper bound $\sup X$. Then, we define infinite sums by setting:

$$\bigoplus_{x \in X} x \stackrel{\text{def}}{=} \sup X \ .$$

We say that an isotone[3] mapping f from a complete idempotent monoid E to a complete idempotent monoid F is *lower semicontinuous*[4] (for short, l.s.c.) [1] if for all (possibly infinite) nonempty subsets $X \subset E$,

$$f\left(\bigoplus_{x \in X} x\right) = \bigoplus_{x \in X} f(x) \ . \tag{3}$$

A dioid is *complete* whenever its underlying additive monoid is complete, and when for all $a \in \mathcal{D}$, the operators of right and left multiplication by a, $x \mapsto xa$, and $x \mapsto ax$ are l.s.c.

Example 1. The dioid $\mathbb{R}_{\max}$ which is not complete can be embedded in the complete dioid $\overline{\mathbb{R}}_{\max} = (\mathbb{R} \cup \{\pm\infty\}, \max, +)$, with the convention $-\infty + (+\infty) = -\infty$.

[3]A mapping $f : (\mathcal{E}, \leq) \to (\mathcal{F}, \leq)$ is *isotone* if it is a morphism of ordered sets, i.e. if $x \leq y \implies f(x) \leq f(y)$.

[4]The specialization of this definition to isotone mappings $\mathbb{R} \to \mathbb{R}$ coincides with the usual lower semicontinuity notion. Lower semicontinuity can also be seen as a special case of Scott continuity defined for (possibly non isotone) mappings over *continuous* lattices. See [8].

2.2 Residuation

A mapping f from an ordered set $(\mathcal{E}, \leq)$ to an ordered set $(\mathcal{F}, \leq)$ is *residuated* if it is isotone, and if for all $y \in F$, the set $\{x \in \mathcal{E} \mid f(x) \leq y\}$ admits a maximal element, denoted $f^\sharp(y)$. The isotone mapping $f^\sharp : (\mathcal{F}, \leq) \to (\mathcal{E}, \leq)$ is called the *residual* of f. The residual $f^\sharp$ is the only isotone mapping satisfying the following relations[5]:

$$f \circ f^\sharp \leq I \ , \tag{4a}$$
$$f^\sharp \circ f \geq I \ . \tag{4b}$$

A simple characterization holds in the case of *complete* idempotent monoids $\mathcal{E}, \mathcal{F}$. Then, f is residuated iff f is l.s.c.[6] and $f(\varepsilon) = \varepsilon$. The following identities can be easily derived from (4).

$$f \circ f^\sharp \circ f = f \ , \tag{5a}$$
$$f^\sharp \circ f \circ f^\sharp = f^\sharp \ , \tag{5b}$$
$$(f \circ h)^\sharp = h^\sharp \circ f^\sharp \ , \tag{5c}$$

where f, h are residuated mappings, $f : \mathcal{E} \to \mathcal{F}$, $h : \mathcal{H} \to \mathcal{E}$, respectively.

The notion of *dually residuated* mapping is defined naturally by reversing the order in the above definitions. See [1] for details. We use the notation $f^\flat$ for the dual residual of f. An immediate consequence of characterization (4) and its dual is that a residuated map $f^\sharp$ is itself dually residuated. Indeed:

$$\left(f^\sharp\right)^\flat = f \ . \tag{6}$$

3 Kernels, images, and factorization of isotone maps

Definition 1 (Kernel). Let C denote a mapping[7] $\mathcal{X} \to \mathcal{Y}$. We call *kernel* of C (denoted $\ker C$), the equivalence relation over $\mathcal{X}$:

$$x \stackrel{\ker C}{\sim} y \Leftrightarrow C(x) = C(y) \ . \tag{7}$$

We will write

$$[x]_C = x \oplus \ker C \stackrel{\text{def}}{=} C^{-1}(C(x))$$

for the equivalence class of x. The notation $x \oplus \ker C$, (to be used when $\mathcal{X}, \mathcal{Y}$ are equipped with additive structures), is introduced by analogy with conventional linear algebra,

[5]We denote I the identity map, without reference to the underlying set, which should be clear from the context. E.g., in (4a), I stands for the identity map $\mathcal{F} \to \mathcal{F}$.

[6]All the examples of linear operators we have in mind for control purposes, and in particular general input-output operators with kernel representation, as in [1, Th.6.5], are l.s.c., and therefore, residuated.

[7]We need not restrict the definition of kernels to morphisms as usual [3]. Indeed, definition (7) is a purely set-theoretic one.

although a kernel is not[8] a 'subspace' of $\mathcal{X}$. It may be thought of as a 'fibration' of $\mathcal{X}$ by the equivalence classes. Even when C is linear, these equivalence classes may have no uniform 'dimension' in that some may be reduced to singletons whereas other may contain infinitely many elements (see the examples in §7).

Lemma 1. *Let $C : \mathcal{X} \to \mathcal{Y}$ be a residuated (resp. dually residuated) mapping. Then* $\ker C = \ker(C^\sharp \circ C)$ *(resp.* $\ker C = \ker(C^\flat \circ C)$*).*

Proof: By this statement, we mean that the equivalence classes are the same with both mappings. We must prove that

$$C(x) = C(y) \Leftrightarrow C^\sharp \circ C(x) = C^\sharp \circ C(y) \ ,$$

for all x, y, which follows from (5a). The same for $^\flat$ instead of $^\sharp$. ∎

Remark 1. In conventional algebra, if C and D are linear operators, $\ker(D \circ C) = \ker C$ iff $\operatorname{im} C \cap \ker D = \{0\}$. Therefore, here, we may say that $\operatorname{im} C$ is 'transverse' to $\ker C^\sharp$ (which is reminiscent of the fact that $\operatorname{im} C$ is orthogonal to $\ker C^T$ — ^{T} denotes transposition — for matrices in conventional algebra). A more accurate way of saying this is to say that the intersection of any equivalence class defined by $\ker C^\sharp$ (or $\ker C^\flat$) with $\operatorname{im} C$ is reduced to a singleton. Indeed, the previous proof establishes uniqueness (it may be read as follows: if two elements are both equivalent $\mod \ker C^\sharp$ and belong to $\operatorname{im} C$, they are equal). To prove that there exists at least one element in every class which lies also in $\operatorname{im} C$, it suffices to observe that, for any x, $C \circ C^\sharp(x) \in x \oplus \ker C^\sharp$ (since $C^\sharp \circ C \circ C^\sharp(x) = C^\sharp(x)$), and that, in addition, it belongs to $\operatorname{im} C$. This operator $C \circ C^\sharp$ will later on be denoted Π_C and called 'least projector onto $\operatorname{im} C$'.

Identifying the equivalence relation $\overset{\ker C}{\sim}$ with its graph $\{(x, y) \mid x \overset{\ker C}{\sim} y\}$, we naturally order kernels by inclusion. This allows us to state the following isotone version of a familiar result for vector spaces.

Lemma 2. *Consider an isotone mapping $G : \mathcal{X} \to \mathcal{G}$, together with a residuated mapping $F : \mathcal{X} \to \mathcal{F}$. The following conditions are equivalent*

1. $\ker F \subset \ker G$
2. *there exists an isotone mapping $H :$ such that $G = H \circ F$*
3. $G = G \circ F^\sharp \circ F$.

Proof: $3 \Longrightarrow 2 \Longrightarrow 1$ is straightforward. We prove that $1 \Longrightarrow 3$. By (5a), we have $(x, F^\sharp \circ F(x)) \in \ker F \subset \ker G$, hence $G(x) = G \circ F^\sharp \circ F(x)$. ∎

[8] The linearity of C is reflected by the fact that $\overset{\ker C}{\sim}$ is a *congruence*, that is for all $x, x', y \in \mathcal{X}$ and for all scalars λ, $x \overset{\ker C}{\sim} x' \implies x \oplus y \overset{\ker C}{\sim} x' \oplus y$, and $x \overset{\ker C}{\sim} x' \implies \lambda x \overset{\ker C}{\sim} \lambda x'$.

For the sake of symmetry, we give the dual results for images. Given a *mapping*[9] $B : \mathcal{U} \to \mathcal{X}$, we define as usual $\operatorname{im} B = \{B(u) \mid u \in \mathcal{U}\}$.

Lemma 3. *Let $B : \mathcal{U} \to \mathcal{X}$ be a residuated (resp. dually residuated) mapping. Then* $\operatorname{im} B = \operatorname{im}(B \circ B^\sharp)$ *(resp.* $\operatorname{im} B = \operatorname{im}(B \circ B^\flat)$*).*

Proof: When B is residuated, this follows readily from (5a). Dual argument for $B^\flat$. ∎

Lemma 4. *Consider an isotone mapping $F : \mathcal{F} \to \mathcal{X}$, together with a residuated mapping $G : \mathcal{G} \to \mathcal{X}$. The following conditions are equivalent*

1. $\operatorname{im} F \subset \operatorname{im} G$
2. *there exists an isotone mapping $H : \mathcal{F} \to \mathcal{G}$ such that $F = G \circ H$*
3. $F = G \circ G^\sharp \circ F$.

Proof: $3 \Longrightarrow 2 \Longrightarrow 1$ is straightforward. We prove that $1 \Longrightarrow 3$. Indeed, for all $x \in \mathcal{F}$, there exists $y \in \mathcal{G}$ such that $F(x) = G(y)$. Therefore, by (5a), $G \circ G^\sharp \circ F(x) = G \circ G^\sharp \circ G(y) = G(y) = F(x)$. ∎

Remark 2. Let us write $G \equiv_{\mathcal{L}} F$ if $G = H \circ F$ and $F = H' \circ G$ for some residuated mappings H, H'. By Lemma 2, two isotone mappings F and G have the same ker iff $G = H \circ F$ and $F = H' \circ G$ for some residuated mappings H, H', which allows us to identify kernels to equivalence classes modulo the left *Green relation* $\equiv_{\mathcal{L}}$ [11]. Dually, defining the right Green relation $G \equiv_{\mathcal{R}} F$ iff $G = F \circ H$ and $F = G \circ H'$ for some isotone mappings H, H', we may identify images with equivalences classes modulo $\equiv_{\mathcal{R}}$.

Remark 3. The interest of the last statement in Lemma 2 and 4 is the effectivity. When dealing with linear mappings over $\mathbb{R}^n_{\max}$, residuals are easily computed (see [5] and [1, Lemma 4.83]). Thus, the inclusion and equality of kernels and images can be effectively checked. Note that in the (max, +) case, by composition of linear mappings and their residuals, we obtain special cases of '(min, max)' homogeneous mappings studied by Olsder and Gunawardena (see [10]).

Remark 4. When F, G are linear operators, one may naturally ask for analogues of Lemma 2 and 4 *restricted* to linear mappings (the isotone mappings $H = G \circ F^\sharp$ or $H = G^\sharp \circ F$ are in general nonlinear). Observing that the restriction to $\operatorname{im} F$ of a mapping H satisfying condition 2 of Lemma 2 is linear, we see that one part of the problem is equivalent to extending a linear mapping defined on a subspace to a globally defined linear mapping. See [7, Ch.0,Th.7.1.1] for a particular case of this result.

[9] We do not reserve the term image to morphisms.

4 Projection parallel to the kernel of an operator

From Remark 1 (or rather, its dual), for any x, there exists a single element in $(x \oplus \ker C) \cap \operatorname{im} C^\sharp$ and it is given by $C^\sharp \circ C(x)$. The following lemma gives other properties of this element.

Lemma 5. *Let $C : \mathcal{X} \to \mathcal{Y}$ be a residuated mapping and let $\Pi^C = C^\sharp \circ C$. We have that*

1. *Π^C is a projector, i.e. $\Pi^C \circ \Pi^C = \Pi^C$;*
2. *$\Pi^C \geq I$ (identity over $\mathcal{X}$);*
3. *$\Pi^C(x)$ is the unique element equivalent to $x \mod \ker C$ which also lies in $\operatorname{im} C^\sharp$;*
4. *$\Pi^C(x)$ is the greatest element in the equivalence class of x;*
5. *$C \circ \Pi^C = C$;*
6. *C is injective iff $\Pi^C = I$ and iff $C^\sharp$ is surjective.*

Proof: The first two statements follow from (5a) and (4b). The third one was already explained in Remark 1. As for the fourth one, it is also a direct consequence of residuation theory: for any x, let $y \in x \oplus \ker C$, hence $C(y) = C(x)$; the greatest y which satisfies such an equation is, by definition, $C^\sharp\big(C(x)\big)$. The fifth statement is another well known formula (it expresses that $\Pi^C(x) \in x \oplus \ker C$). Finally, the last statement is extracted from [1, Theorem 4.56]. ■

Remark 5. From this lemma, we may call Π^C 'the greatest projector parallel to $\ker C$', or, alternatively, 'the projector onto $\operatorname{im} C^\sharp$ parallel to $\ker C$'. In [1, Definition 4.58], an operator satisfying the first two statements of Lemma 5 was called a *closure mapping*. Suppose C itself is a closure mapping which is, in addition, residuated. Then, according to [1, Theorem 4.59], $\Pi^C = C$.

Remark 6. Needless, to say, if C is dually residuated, dual statements of Lemma 5 can be made: for example, $C^\flat \circ C(x)$ is the unique element of $x \oplus \ker C$ which lies at the same time in $\operatorname{im} C^\flat$, and also the *least* element in $x \oplus \ker C$.

5 Projection onto the image of an operator

As noticed in Remark 1, for any residuated operator $B : \mathcal{U} \to \mathcal{X}$ and any $x \in \mathcal{X}$, $B \circ B^\sharp(x)$ is the unique element of $\operatorname{im} B$ which is equivalent to $x \mod \ker B^\sharp$. By applying what was said in Remark 6 to the dually residuated operator $B^\sharp$, it may be seen that $B \circ B^\sharp(x)$ is also the least element in $x \oplus \ker B^\sharp$. The following lemma adds the interpretation that, among all isotone operators M which preserve $\operatorname{im} B$, that is, $M \circ B(x) = B(x)$ for all $x \in \mathcal{U}$, $B \circ B^\sharp$ is the least one (in addition, it is a projector).

Lemma 6. *Let $B : \mathcal{U} \to \mathcal{X}$ be a residuated mapping and let $\Pi_B = B \circ B^\sharp$.*[10] *We have that*

1. *Π_B is a projector;*
2. *$\Pi_B \leq I$;*
3. *$\Pi_B(x)$ is the unique element equivalent to $x \mod \ker B^\sharp$ which also lies in $\operatorname{im} B$;*
4. *$\Pi_B(x)$ is the least element in the equivalence class of $x \mod \ker B^\sharp$;*
5. *Π_B is the least operator such that $\Pi_B \circ B = B$;*
6. *B is surjective iff $\Pi_B = I$ and iff $B^\sharp$ is injective.*

Proof: Given that the other statements can be easily derived from Lemma 3 and (4), (5) or have already been established, only statement 5 needs some argument. Indeed, this statement will be reformulated in another way in the next lemma, and then proved. ■

Lemma 7. *Let B be a residuated mapping from $\mathcal{U}$ to $\mathcal{X}$ and R_B ([11]) be the operator, defined over isotone mappings over $\mathcal{X}$, which associates the mapping $M \circ B$ with M. Then R_B is dually residuated*[12] *and $(R_B)^\flat$ is equal to $R_{B^\sharp}$.*

Proof: By definition, $R_B^\flat(\Psi)$ is the least operator M such that

$$M \circ B \geq \Psi \; . \tag{8}$$

Therefore, since $B \circ B^\sharp \leq I$,

$$M \geq M \circ B \circ B^\sharp \geq \Psi \circ B^\sharp \; .$$

Moreover, $\Psi \circ B^\sharp$ itself satisfies (8) since $B^\sharp \circ B \geq I$. Hence, it is the solution to the dual residuation problem. ■

Remark 7. From Lemma 6, we may call Π_B 'the least projector onto $\operatorname{im} B$', or, alternatively, 'the projector onto $\operatorname{im} B$ parallel to $\ker B^\sharp$', or even 'the least projector parallel to $\ker B^\sharp$. In [1, Definition 4.58], an operator satisfying the first two statements of Lemma 6 was called a *dual closure mapping*. However, if B itself is a dual closure mapping which is *residuated* — and not *dually* residuated —, it does not seem possible to state in general that $\Pi_B = B$.

Remark 8. If B is dually residuated, dual statements of Lemma 6 can be made: for example, $B \circ B^\flat(x)$ is the unique element of $x \oplus \ker B^\flat$ which lies at the same time in $\operatorname{im} B$, and also the *greatest* element in $x \oplus \ker B^\flat$, and $B \circ B^\flat$ is the *greatest* operator which preserves $\operatorname{im} B$.

[10] Note that B as a subscript refers to the expression $B \circ B^\sharp$ whereas B as a superscript refers to $B^\sharp \circ B$.

[11] R for composition to the 'right'

[12] It is also residuated but we do not have a closed-form expression for $(R_B)^\sharp$ since B is not necessarily dually residuated.

6 Projection on the image of an operator parallel to the kernel of another operator

6.1 Discussion

Let $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ be three ordered sets and $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be two isotone operators. Given any $x \in \mathcal{X}$, we now raise the problem of finding $y \in \operatorname{im} B \cap (x \oplus \ker C)$, that is,

$$\text{find } y \in \mathcal{X},\ \text{s.t. } \exists z \in \mathcal{U} :\ C(y) = C(x)\ , \tag{9a}$$
$$B(z) = y\ . \tag{9b}$$

If such a y exists and is unique, it will be called the projection of x onto $\operatorname{im} B$ parallel to $\ker C$. This, in turn, may raise the problem of existence (is $\operatorname{im} B \cap (x \oplus \ker C)$ nonempty?) or uniqueness (is $\operatorname{im} B \cap (x \oplus \ker C)$ reduced to a singleton?). It is known that residuation theory is a way around the problems of nonexistence (by relaxing equalities to inequalities) and of nonuniqueness (by looking for some 'extremal' — either greatest or least — solution), provided that the direction of inequalities be consistent with the notion of extremality chosen (greatest 'subsolution' or least 'supersolution') and that the operators involved have consistent residuation properties (see [1, §4.4.2]). Also, if equalities can finally be satisfied, residuation will always provide an answer with equalities holding true.

Linear l.s.c. operators are residuated. Therefore, it is justified to privilege the theory in which $B^\sharp$ and $C^\sharp$ do exist. This is what we do hereafter. The dual situation when B and C are dually residuated can be studied similarly.

6.2 Existence

The following lemma gives several necessary and sufficient conditions for the fact that $\operatorname{im} B$ 'crosses' $\ker C$.

Lemma 8. *Let $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be two residuated operators. Let*

$$\Pi_B^C = B \circ (C \circ B)^\sharp \circ C\ . \tag{10}$$

The following statements are all equivalent:

1. *for all $x \in \mathcal{X}$, there exists an element $y \in \operatorname{im} B \cap (x \oplus \ker C)$ and $\Pi_B^C(x)$ is such a y;*
2. *for all $x \in \mathcal{X}$ and $w \in \mathcal{Y}$ such that $w = C(x)$, there exists a $z \in \mathcal{U}$ such that $w = C \circ B(z)$;*
3. $\operatorname{im}(C \circ B) = \operatorname{im} C$;
4. $\Pi_{C \circ B} = \Pi_C$, *that is,* $(C \circ B) \circ (C \circ B)^\sharp = C \circ C^\sharp$;
5. $C \circ \Pi_B^C = C$, *that is,* $C \circ B \circ (C \circ B)^\sharp \circ C = C$.

Proof:

$1 \Rightarrow 2$: Item 2 is a rephrasing of item 1 with $w = C(y)$.

$2 \Rightarrow 3$: Straightforward.

$3 \Rightarrow 4$: Straightforward by recalling that, e.g., Π_C is the least projector onto $\operatorname{im} C$.

$4 \Rightarrow 5$: It suffices to post-compose with C to pass from the equality in 4 to the equality in 5.

$5 \Rightarrow 1$: By application to x, the equality in 5 says nothing but 1. ■

6.3 Uniqueness

The following lemma gives several necessary and sufficient conditions for the fact that $\operatorname{im} B$ 'crosses' $\ker C$ at at most one point.

Lemma 9. *Let $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be two residuated operators. The following statements are all equivalent:*

1. *for all $x \in \mathcal{X}$, there exists at most one element in* $\operatorname{im} B \cap (x \oplus \ker C)$*;*
2. *for all $z, z' \in \mathcal{U}$ such that $C \circ B(z) = C \circ B(z')$, we have $B(z) = B(z')$;*
3. $\ker(C \circ B) = \ker B$*;*
4. $\Pi^{C \circ B} = \Pi^B$, *that is,* $(C \circ B)^\sharp \circ (C \circ B) = B^\sharp \circ B$*;*
5. $\Pi_B^C \circ B = B$, *that is,* $B \circ (C \circ B)^\sharp \circ C \circ B = B$.

Proof:

$1 \Rightarrow 2$: Item 2 is a rephrasing of item 1.

$2 \Rightarrow 3$: Straightforward.

$3 \Rightarrow 4$: Straightforward by recalling that, e.g., Π^B is the greatest projector parallel to $\ker B$.

$4 \Rightarrow 5$: It suffices to pre-compose with B to pass from the equality in 4 to the equality in 5.

$5 \Rightarrow 1$ (or 2): If $C \circ B(z) = C \circ B(z')$, apply $B \circ (C \circ B)^\sharp$ to both members and conclude, using the assumption. ■

6.4 Summary

We summarize the above results as follows.

Theorem 1. *Consider two residuated operators*

$$\mathcal{U} \xrightarrow{B} \mathcal{X} \xrightarrow{C} \mathcal{Y}\ . \tag{11}$$

There exists a unique projection operator Π_B^C (on $\operatorname{im} B$ *parallel to* $\ker C$*) iff conditions of Lemma 8 and 9 above hold true. Then, Π_B^C is given by (10). Equivalently:*

$$\Pi_B^C = \Pi_B \circ \Pi^C \ . \tag{12}$$

Moreover, if B and C are linear, then Π_B^C is linear.

Proof: The only points to check are: (i) factorization (12) which follows from (10) and (5c), (ii) the linearity of Π_B^C, which follows from the linearity of the defining relations (9). ■

Remark 9. When the existence and uniqueness conditions are satisfied, (12) shows that projecting onto $\operatorname{im} B$ parallel to $\ker C$ amounts to projecting onto $\operatorname{im} C^\sharp$ parallel to $\ker C$ first, and then, to project this element onto $\operatorname{im} B$ parallel to $\ker B^\sharp$. Recall that $\ker(C^\sharp \circ C) = \ker C$ and that $\operatorname{im} B = \operatorname{im}(B \circ B^\sharp)$. We thus might have replaced from the beginning C, resp. B, by $C^\sharp \circ C$, resp. $B \circ B^\sharp$, but these operators are obviously neither residuated nor dually residuated.

Remark 10. Note also that, in general,

$$\Pi_B \leq \Pi_B^C \leq \Pi^C \ ,$$

confirming the extremality of Π_B and Π^C observed earlier. These two projectors are less — for the former — and greater — for the latter — than identity, whereas Π_B^C is not comparable to identity in general.

6.5 Duality

Finally, we mention the following useful duality result.

Theorem 2 (Duality). *Consider two residuated mappings B, C as in (11).*

1. *The existence of a projection onto* $\operatorname{im} B$ *parallel to* $\ker C$ *is equivalent to the uniqueness of the projection onto* $\operatorname{im} C^\sharp$ *parallel to* $\ker B^\sharp$*.*
2. *The uniqueness of the projection onto* $\operatorname{im} B$ *parallel to* $\ker C$ *is equivalent to the existence of a projection onto* $\operatorname{im} C^\sharp$ *parallel to* $\ker B^\sharp$*.*

Proof: 1. Using Lemma 8 (item 4) and the dual of Lemma 9 (item 4) (stated for dually residuated mappings), we write the existence condition of a projection onto $\operatorname{im} B$ parallel to $\ker C$ and the uniqueness condition of the projection onto $\operatorname{im} C^\sharp$ parallel to $\ker B^\sharp$, respectively, as follows:

$$C \circ C^\sharp = C \circ B \circ B^\sharp \circ C^\sharp \ , \tag{13a}$$

$$\left(C^\sharp\right)^\flat \circ C^\sharp = \left(C^\sharp\right)^\flat \circ \left(B^\sharp\right)^\flat \circ B^\sharp \circ C^\sharp \ . \tag{13b}$$

Using (6), we see that (13a) and (13b) coincide, which shows the equivalence of the two conditions stated in item 1. The proof for item 2 is similar. ■

Remark 11. Observe that the theory with *dually* residuated operators applies for $\Pi_{C^\sharp}^{B^\sharp}$, and that the dual formula of (10) yields

$$\Pi_{C^\sharp}^{B^\sharp} = C^\sharp \circ C \circ B \circ B^\sharp = \Pi^C \circ \Pi_B \ , \tag{14}$$

which should be compared with (12).

Remark 12. Formulæ (12) and (14) give Π_B^C and $\Pi_{C^\sharp}^{B^\sharp}$ as a function of Π^C and Π_B. We note that, conversely:

$$\Pi_B = \Pi_B^C \circ \Pi_{C^\sharp}^{B^\sharp} \ ,$$
$$\Pi^C = \Pi_{C^\sharp}^{B^\sharp} \circ \Pi_B^C \ .$$

7 Illustrative examples

We start by observing that, if B is linear, $\operatorname{im} B$ is invariant by translation along the vector $\mathbf{1} = \begin{pmatrix}1 & 1 & 1 & \ldots\end{pmatrix}^T$. Indeed, for all $\lambda \in \mathbb{R}$, we have:

$$x \in \operatorname{im} B \Leftrightarrow x + \lambda \mathbf{1} \in \operatorname{im} B \ ,$$

where (exceptionally), the operations have to be interpreted in conventional algebra. Likewise, if C is linear, the equivalence classes defined $\ker C$ are 'reproducible' by translations along $\mathbf{1}$ in the sense that

$$(x, y) \in \ker C \Leftrightarrow (x + \lambda \mathbf{1}, y + \lambda \mathbf{1}) \in \ker C \ .$$

Therefore,in the following examples, we can limit ourselves to determine enough classes to fill in the whole space by these translations.

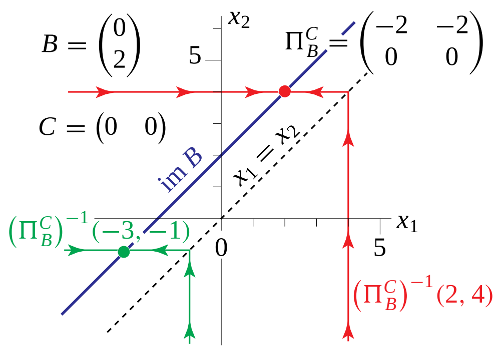

Figure 1: Projection on a line parallel to an hyperplane

Example 2. Let $\mathcal{X} = \mathbb{R}^2_{\max}$, $\mathcal{U} = \mathbb{R}_{\max}$, $\mathcal{Y} = \mathbb{R}_{\max}$, and consider the two linear mappings with respective matrices:

$$B = \begin{pmatrix}a\\b\end{pmatrix} \ , \quad C = \begin{pmatrix}c & d\end{pmatrix} \ . \tag{15}$$

As soon as $ac \oplus bd \neq \varepsilon$, CB is invertible, and the existence/uniqueness conditions are trivially satisfied[13]. A generic example is displayed in Fig 1. The image of B is the conventional line crossing the point $(0, 2)$ as shown in the figure. The two broken lines (with arrows) represent the preimage by Π_B^C of the two points (bold circles) $(2, 4)$ and $(-3, -1)$, respectively.

[13] The example extends immediately to $\mathcal{X} = \mathbb{R}^n_{\max}$, when $\operatorname{im} B$ is a line (i.e. when B has only one column) and $\ker C$ is an hyperplane (when C has only one row).

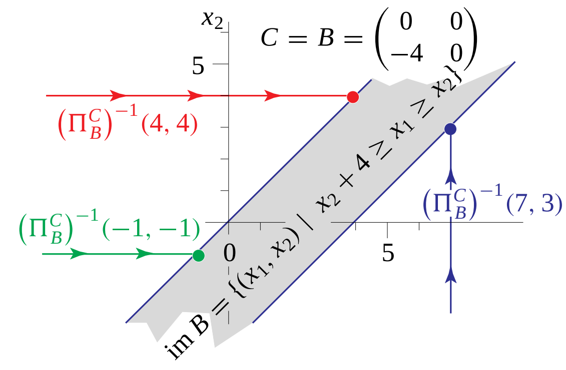

Figure 2: Projection on a maximal rank strict subspace

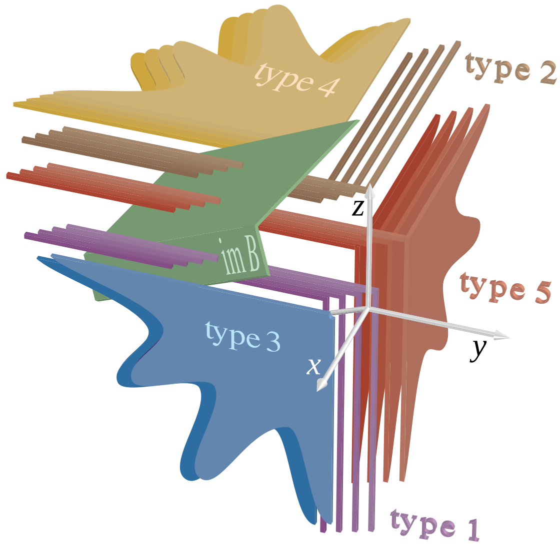

Figure 3: 3-dimensional case

Example 3. Consider the matrices

$$B = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, \quad C = B \; .$$

Note that $CB = B$ is not invertible. Note also that the columns of B span a *strict* subspace of $\mathbb{R}^2_{\max}$, with 'rank' (minimal number of generators) 2: such a situation cannot occur in conventional algebra. However, the projection problem admits a unique solution depicted in Fig. 2. Three types of classes are shown. If $x_2 + 4 > x_1 > x_2$, then x lies in the interior of $\mathrm{im}\,B$. The equivalence class $[x]_C = C^{-1}\big(C(x)\big)$ is reduced to $\{x\}$ itself, and thus $\Pi_B^C(x) = x$. If $x_2 \geq x_1$ (x is above the upper boundary of $\mathrm{im}\,B$), $[x]_C = \{(t, x_2) \mid t \leq x_2\}$ is an horizontal half line crossing $\mathrm{im}\,B$ at the unique point (x_2, x_2), that is, $\Pi_B^C(t, x_2) = (x_2, x_2)$ for $t \leq x_2$. Dually, the points below the lower boundary of $\mathrm{im}\,B$ are projected on this boundary along vertical half lines, as shown in Fig 2.

Example 4. Ex. 3 is a special case of the following. Consider an arbitrary idempotent matrix P (i.e. $P = P^2$). We note that $\Pi_P^P = P(P^2)^\sharp P = PP^\sharp P = P$ (by (5a)), and that $\Pi_P^P P = P \Pi_P^P = P$. Thus, $P = \Pi_P^P$ is the unique projector on $\mathrm{im}\,P$ parallel to $\ker P$. This result can be interpreted as an analogue of the familiar fact that with a conventional idempotent matrix P is associated the projection on $\mathrm{im}\,P$ parallel to the supplementary space $\mathrm{im}\,(I - P)$.

Remark 13. Conversely, when $\mathcal{X} = \mathcal{D}^n$ is the free semimodule with n generators over a dioid $\mathcal{D}$, and when C, B are *linear* operators satisfying the existence and uniqueness condition of Theorem 1, Π_B^C is a linear idempotent operator $\mathcal{D}^n \to \mathcal{D}^n$. Therefore, it is represented by an idempotent matrix P in the canonical basis.

Example 5. Consider the matrices

$$B = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} .$$

The image of B and the equivalences classes modulo C are represented in Fig. 3. Let us detail the construction of this picture.

Then, it is not difficult to see that there are exactly 5 types of equivalence classes, namely:

$$[(\alpha, 1, 1)^T]_C = \{(\alpha, s, t)^T \mid \max(s,t) = 1\} \quad \text{with } 0 < \alpha < 1 \quad \text{(type 1)},$$

$$[(0, 1, \alpha)^T]_C = \{(s, t, \alpha)^T \mid \max(1+s, t) = 1\} \quad \text{with } 1 < \alpha < 2 \quad \text{(type 2)},$$

$$[(1, 1, 1)^T]_C = \{(1, s, t)^T \mid s, t \leq 1\} \quad \text{(type 3)},$$

$$[(0, 1, 2)^T]_C = \{(s, t, 2)^T \mid s \leq 0, t \leq 1\} \quad \text{(type 4)},$$

$$[(0, 1, 1)^T]_C = \{(s, 1, t)^T \mid s \leq 0, t \leq 1\} \cup \{(0, s, 1)^T \mid s \leq 1\} \quad \text{(type 5)}.$$

Modulo the translations along $\mathbf{1}$, the last three types are unique classes whereas type 1 and 2 classes fill in the gaps left between the previous classes: this is achieved by letting the parameter α vary within the given bounds. It should be clear (by mere inspection of the picture) that each equivalence class modulo C crosses $\mathrm{im}\,B$ at exactly one point. Therefore, the 'direct sum' conditions of Theorem 1 are satisfied. We may of course prove this algebraically by checking the conditions expressed in Lemma 8 (item 5) and Lemma 9 (item 5), but Fig. 3 is probably more informative.

Remark 14. Consider the case where $\mathcal{X} = \mathbb{R}^n_{\max}$ and B and C are linear. A necessary condition for the projection Π_B^C to exist is that any *basis* [15] of $\mathrm{im}\,B$ has at most n generators[14]. Indeed, when it exists, Π_B^C is a linear operator from $\mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$ with image $\mathrm{im}\,B$, but the image of such an operator is generated by at most n elements. Since a basis of $\mathrm{im}\,B$ is obtained by eliminating elements from an arbitrary generating family [15], the necessary condition is proved.

[14] As shown in [5], a basis of a finitely generated subspace of $\mathbb{R}^n_{\max}$ (with $n \geq 3$) may have arbitrarily many elements.

Remark 15. Using the duality theorem 2, we obtain another obstruction dual to that stated in Remark 14: for the projection Π_B^C to exist, it is necessary that any *basis* of $\operatorname{im} C^\sharp$, seen as a semimodule over the dual $\overline{\mathbb{R}}_{\min} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{\pm\infty\}, \min, +)$ semiring, has at most n elements.

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Exemple 3D du papier WODES'96:

$$B\pm C\pm B/\ \pm C \qquad \begin{matrix} 0 & 1 & 2 \\ 1 & 1\ 5 & 1\ 5 \\ 0 & 0 & 0 \end{matrix}$$

source :

```
B\comp(C\comp B)^{\sharp}\comp C=
\begin{pmatrix}
0&-1&-2\\-1&-1.5&-1.5\\0&0&0
\end{pmatrix}
```