Contribution of Stochastic Control Singular Perturbation Averaging and Team Theories to an Example of Large-Scale Systems: Management of Hydropower Production

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***Abstract*—We present a global model describing a hydropower production system and the related management problem. Using averaging and singular perturbation techniques, we define a nearby optimal problem. The optimization in the class of local feedbacks leads to a team problem which can be solved numerically.**

Introduction

We describe a hydropower system and the related operational optimization problem. The stochastic and nonlinear characteristics of this problem impose the dynamic programming approach, which leads to a dimensionality problem. We have to solve a partial differential equation on R^{200}.

To overcome this difficulty, we use two kinds of techniques.

1) The first one uses a novel singular perturbation and averaging technique to define a relevant simplified problem.

2) The second one consists of constraining the class of admissible strategies to a class of local feedbacks which keep the dynamics of the system uncoupled.

At the end of this process, we have to control a system of about 20 two-dimensional partial differential equations. This problem, which may be viewed as a team problem, is solved numerically on a test example. Then, a local problem (management of a valley) is solved on real data.

The mathematical techniques are explained for specific examples, and afterwards applied to the hydropower system.

The plan will be the following.

I. The Hydropower System

A. The description of a valley
B. The model of water inputs
C. The evolution of the stocks of water
D. The demand of electricity
E. The cost function
F. The management problem
G. A change of variables
H. The dynamic programming equation
I. The limit problem when $\epsilon \to 0$.

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J. A relevant simplified problem
K. A team problem
L. Approximation of multiple integrals
M. A numerical example.

II. NUMERICAL SOLVING FOR A REAL VALLEY

A. Identification of water inputs
B. Bellman's equation
C. Discretization of Bellman's equation.

APPENDIX I

A periodic dynamic programming equation

APPENDIX II

An averaging singular perturbation problem
1) The averaging theorem
2) Interpretation of the averaging theorem

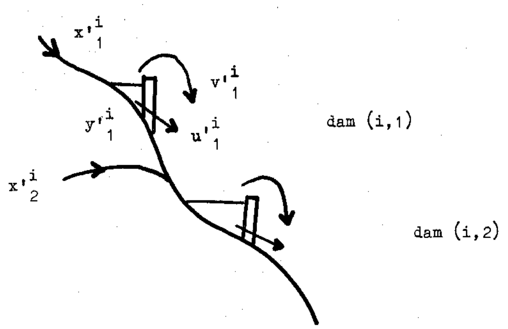
I. THE HYDROPOWER SYSTEM

We shall describe a model which wants to be relevant for a one-year management of a hydropower system. Its purpose is to give a good idea of the weekly outputs of the big seasonal dams.

We consider $I(\sim 20)$ hydroelectric valleys. Each valley is equipped with $J_i(\sim 10)$ dams in cascade. This assumption is a simplification of the real system: in fact, we need only to be able to divide the system into subsystems with independent dynamics. Each subsystem (here one valley) must be computable after simplification (here particular treatment of "small" dams) by the dynamic programming approach.

A. The Description of a Valley

Let us consider the ith valley:



where x'^i_j is the inflow of water between dam $(i,j-1)$ and dam (i,j).

For the dam (i,j) we denote

y'^i_j the stock of water
$\check{y}^i_j$ the capacity of storage
u'^i_j the water release through the turbines
$\check{u}^i_j$ the maximal release through the turbines
v'^i_j the water release through spillway
$e'^i_j(y'^i_j, u'^i_j)$ the power generated when the stock is y'^i_j and the turbined release is u'^i_j.

B. The Modeling of Water Inputs

The inputs of water being very uncertain, a deterministic model is irrelevant. We model the inputs of water by independent diffusion processes. By this way we obtain all Markov processes of dimension 1 with continuous trajectories. The assumption of independence is not very realistic, but it is necessary to numerically solve the problem.

The input in the upper dam is given by

$$dx'^i_1 = b'^i(s, x'^i_1(s))\,ds + \sqrt{a'^i(s, x'^i_1(s))}\;dB^i_s \tag{1.1}$$

where b'^i and a'^i are to be identified ([30], [24], [18]–[20] and Section II) and B^i $i = 1 \cdots I$ are independent Brownian motions.

We suppose that

$$x'^i_j(s) = f'^i_j(s, x'^i_1(s)), \qquad j = 2 \cdots J_i,\ i = 1 \cdots I. \tag{1.2}$$

Equation (1.2) means that intermediary inputs are functionally linked to the input of the upper dam.

Moreover, we suppose that the functions $s \to (b'^i(s,x), a'^i(s,x), f'^i_j(s,x))$ are periodic functions with a period of one year.

C. The Evolution of the Stocks of Water

The variation of the stock of water is equal to the inputs minus the outputs:

$$dy'^i_j = \left(x'^i_j + u'^i_{j-1} + v'^i_{j-1} - u'^i_j - v'^i_j\right) ds \tag{1.3}$$

with the constraints

$$0 \leq u'^i_j \leq \check{u}^i_j \tag{1.4}$$

$$0 \leq v'^i_j \tag{1.5}$$

$$u'^i_j + v'^i_j \leq x'^i_j + u'^i_{j-1} + v'^i_{j-1}, \qquad \text{if } y'^i_j = 0 \tag{1.6}$$

$$v'^i_j + u'^i_j \geq x'^i_j + u'^i_{j-1} + v'^i_{j-1}, \qquad \text{if } y'^i_j = \check{y}^i_j. \tag{1.7}$$

Equation (1.6) means that when the stock is empty, the output cannot exceed the input.

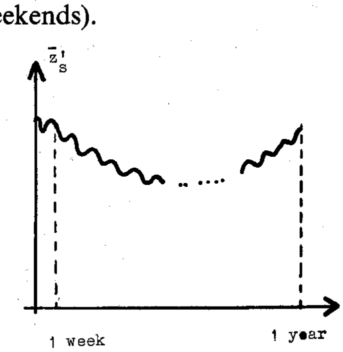
Equation (1.7) means that when the stock is full, the output exceeds the input.

D. The Demand of Electricity

The demand for electrical power z' is modeled by a diffusion process; this process takes into account the "double periodicity" of the demand:

1) one-year oscillation if we neglect the long term trends as climatic or market perturbations

2) one-week oscillation (each working day the demand presents two peaks and there are particular phenomena during the weekends).



We set

$$dz'_s = \frac{1}{\epsilon}\,\beta'\left(s, \frac{s}{\epsilon}, z'_s\right)ds + \sqrt{\frac{\alpha'}{\epsilon}}\; dB_s \qquad (1.8)$$

where $s \to \beta(s,\theta,z')$ and $\theta \to \beta(s,\theta,z')$ are one-year periodic functions and $\epsilon = 1/52$.

Indeed, if we look at the demand of electricity with a step of discretization of one hour, during one week, it presents many perturbations around a deterministic trajectory. If we look at the demand with a step of discretization of one week, during one year, it seems deterministic. The model (1.8) takes into account this phenomena. If we take a unit of time of one week, a diffusion $dz' = \beta'(s,z')ds + \sqrt{\alpha'}\, dB_s$ is good; now if we change the unit of time, we take a unit of one year, and we obtain

$$dz' = \frac{1}{\epsilon}\,\beta'\left(\frac{s}{\epsilon}, z'\right)ds + \sqrt{\frac{\alpha'}{\epsilon}}\; dB_s;$$

and now if we want to take into account the one-year oscillations, we have the model (1.8) where the application $(t,\theta,z) \to \beta(t,\theta,z)$ and α must be identified.

E. The Cost Function

The cost function is the expectation of the cost of meeting the electricity demand over one year.

Let us denote

$$h'(z',y',u') = z' - \sum_i \sum_j e'^{i}_{j}\left(y'^{i}_{j}, u'^{i}_{j}\right); \qquad (1.9)$$

the power produced by nonhydroelectrical means when the demand is z', the stocks y', and the turbine releases u'.

$c'(h')$ is the cost or production of h' per unit of time. It is, in fact, the result of the optimization of nonhydroelectrical means [32], [12].

So, the nonhydropower part of electricity appears only in the cost function. This is also a simplification, but realistic for the purpose of this model.

F. The Management Problem

If we require that all the feedbacks (u',v') are one-year periodic functions, we may suppose that there exists a unique initial law q'_0 for the states (x'_0, y'_0, z'_0) such that (x'_s, y'_s, z'_s) admits a one-year periodic law. Then the management problem consists of minimizing

$$E\int_0^T c'(h')\,ds \qquad (1.10)$$

with respect to all admissible strategies (u',v') where T is the period of management (one year) (the expectation is taken for the process starting with the initial law q'_0). Then, by ergodic consideration, we can prove that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t c'(h')\,ds = (1.10). \qquad (1.11)$$

Remark: The interest of such a criterion is that it does not depend on the starting point. This will be an important point when we constrain the admissible strategies to local feedbacks.

Remark: This problem is studied in [10] by simulation techniques for the French case. In [32] and [36] the Quebec case is studied. In [14] a valley is optimized. In [12] the problem is studied, taking into account the starting cost and the availability of the power plant for the New Caledonia case. In [16] and [17] two intermediary steps of this presentation are given.

G. A Change of Variables

Let us make a change of variables such that all the new ones become dimensionless.

If T = one year, Z = the total installed power, and κ = the cost of production of the power Z during T, let us define

$t = s/T$
$y^i_j = y'^i_j / \check{y}^i_j$
$u^i_j = u'^i_j / \check{u}^i_j$
$v^i_j = v'^i_j / \check{u}^i_j$
$x^i_j = x'^i_j / \check{u}^i_j$
$\xi^i_j = \check{y}^i_j / \check{y}^i_1$
$k^i_j = \check{u}^i_{j-1} / \check{u}^i_j$
$\nu^i_j = T\check{u}^i_j / \check{y}^i_1$
$b^i(t,x^i_1) = Tb'^i(tT, x^i_1\check{u}^i_1)/\check{u}^i_1$
$a^i(t,x^i_1) = Ta'^i(tT, x^i_1\check{u}^i_1)/(\check{u}^i_1)^2$
$f^i_j(t,x^i_1) = f'^i_j(tT, x^i_1\check{u}^i_1)/\check{u}^i_1$
$e''^i_j(y^i_j,u^i_j) = e'^i_j(\xi^i_j y^i_j \check{y}^i_1, u^i_j\check{u}^i_j)/Z$
$z = z'/Z$
$\beta(t,\theta,z) = T\beta'(tT,\theta T, zZ)/Z$
$\alpha = \alpha' T/Z^2$
$c(h) = Tc'(hZ)/\kappa.$

We suppose that the design of a valley is the following:

1) a big seasonal dam at the head of the valley
2) small weekly dams downstream
3) the maximal capacities of turbine plants do not depend on their capacities of storage. (In general, they are of the same order in the same valley in agreement with the fact that the water flow of a river is increasing when the altitude decreases.)

These assumptions are often a good approximation of the reality; if they are not verified (valley in V), we can aggregate several big dams into one in such a way so as to have a good representation of the reality by this structure.

These assumptions are mathematically written by

$$\xi^i_1 = 1; \quad \xi^i_j \sim \epsilon \ll 1,^{1} \qquad j \neq 1;$$

$$\nu^i_j \text{ independent of } \epsilon; \quad \epsilon = \tfrac{1}{52}$$

(1/number of weeks in a year).

From now on we shall distinguish the upper dams from the others. For that, let us denote

$1^i_j = \nu^i_j\epsilon/\xi^i_j, \qquad j \neq 1$
$X^i = x^i_1$
$Y^i = y^i_1$
$U^i = u^i_1$
$V^i = v^i_1$
$L^i = \nu^i_1$
$e^i_j(\epsilon y^i_j, u^i_j) = e''^i_j(y^i_j, u^i_j), \qquad i = 1\cdots I,\ j = 2\cdots J_i,$
$e^i_1 = e''^i_1.$

[1] $\sim$ means same order.

From now on, x(resp.y,u,v) denotes (x_j^i) (resp.(y_j^i), $(u_j^i),v_j^i)$), $i=1\cdots I, j=2\cdots J_i$.

So we have to solve the following stochastic control problem:

$$dX^i = b^i(t,X^i)dt + \sqrt{a^i}\,(t,X^i)dB_t^i \tag{1.14}$$

$$dY^i = L^i(X^i - U^i - V^i)dt, \qquad i=1\cdots I \tag{1.15}$$

$$\epsilon dy_j^i = 1_j^i\big(x_j^i + (1_{j-1}^i + v_{j-1}^i)k_j^i - u_j^i - v_j^i\big)dt,$$

$$i=1\cdots I,\ j=2\cdots J_i \tag{1.16}$$

$$x_j^i = f_j^i(t,X^i) \tag{1.17}$$

$$0 \leqslant u_j^i \leqslant 1 \tag{1.18}$$

$$v_j^i \geqslant 0 \tag{1.19}$$

$$u_j^i + v_j^i \leqslant x_j^i + k_j^i(u_{j-1}^i + v_{j-1}^i), \qquad \text{if } y_j^i = 0 \tag{1.20}$$

$$u_j^i + v_j^i \geqslant x_j^i + k_j^i(u_{j-1}^i + v_{j-1}^i), \qquad \text{if } y_j^i = 1,$$

$$i=1\cdots I,\ j=1\cdots J_i \tag{1.21}$$

$$dz = 1/\epsilon\beta(t,t/\epsilon,z)dt + \sqrt{\alpha/\epsilon}\ d\tilde{B}_t \tag{1.22}$$

$$h(Y,\epsilon y,z,U,u) = z - \sum_i e_1^i(Y^i,U^i) - \sum_i\sum_j e_j^i(\epsilon y_j^i, u_j^i) \tag{1.23}$$

$$\mu^\epsilon = \min_{U,V,u,v} E\int_0^1 c(h)\,dt. \tag{1.24}$$

H. The Dynamic Programming Equation

Let us define the operators

$$A^i(U^i,V^i) = b_8^i(t,X^i)\frac{\partial}{\partial X^i} + \frac{1}{2}a^i(t,X^i)\frac{\partial^2}{\partial X^{i2}} + L^i(X^i - U^i - V^i)\frac{\partial}{\partial Y^i} \tag{1.25}$$

$$F_j^i(u^i,v^i) = 1_j^i\big(x_j^i + (u_{j-1}^i + v_{j-1}^i)k_j^i - u_j^i - v_j^i\big)\frac{\partial}{\partial y_j^i} \tag{1.26}$$

$$D(\theta) = \beta(t,\theta,z)\frac{\partial}{\partial z} + \frac{1}{2}\alpha\frac{\partial^2}{\partial z^2}. \tag{1.27}$$

The solution $\mu^\epsilon, W^\epsilon(t,X,y,z)$ of

$$\begin{cases} \dfrac{\partial W^\epsilon}{\partial t} + 1/\epsilon D(t/\epsilon)W^\epsilon + \min\limits_{U,V,u,v}\Big[\sum\limits_i A^i(U^i,V^i)W^\epsilon \\ \quad + 1/\epsilon \sum\limits_i \sum\limits_{j=2}^{J_i} F_j^i(u^i,v^i)W^\epsilon \\ \quad + c\big(h(Y,\epsilon y,z,U,u)\big)\Big] = 0 \\ W^\epsilon(0,X,Y,y,z) = W^\epsilon(1,X,Y,y,z) + \mu^\epsilon \end{cases} \tag{1.28}$$

gives the optimal cost of the problem (1.14)$\cdots$(1.24), as is shown in Appendix I.

The numerical solution of (1.28) is impossible because the large dimension of the state $I + \Sigma_i J_i + 1 \cong 200$ $((X,Y,y,z) \in R^{200})$.

I. The Limit of the Dynamic Programming Equation when $\epsilon \to 0$

Appendix II gives a singular perturbation theorem. Thanks to this theorem, we know what happens to the dynamic programming equation when $\epsilon \to 0$.

The W_ϵ solution of (1.28) converges to the $\overline{W}$ solution of

$$\begin{cases} \dfrac{\partial \overline{W}}{\partial t} + \min\limits_{U,u,V,v}\Big[\sum\limits_i A^i(U^i,V^i)\overline{W} + \nu(t,X,Y,U,V,u,v)\Big] = 0 \\ \overline{W}(0,X,Y) = \overline{W}(1,X,Y) + \mu \end{cases} \tag{1.29}$$

with the ν *solution of the short run control problem, that is,* t, X, Y, $\overline{U}$, $\overline{V}$, $\bar{u}$, $\bar{v}$ being given, one has to solve the following stochastic control problem where θ is now the time:

$$\nu(t,X,Y,\overline{U},\overline{V},\bar{u},\bar{v}) = \min_{\tilde{U},\tilde{V},\tilde{u},\tilde{v}} E\int_0^1 c0h(Y,0,z_\theta,\overline{U}+\tilde{U}_\theta,\bar{u}+\tilde{u}_\theta)\,d\theta \tag{1.30}$$

subject to the constraints

$$E\int_0^1 \tilde{u}\,d\theta = 0, \quad E\int_0^1 \tilde{v}\,d\theta = 0,$$

$$E\int_0^1 \tilde{U}\,d\theta = 0, \quad E\int_0^1 \tilde{V}\,d\theta = 0 \tag{1.31}$$

$$dy_j^i = 1_j^i\big(x_j^i + (u_{j-1}^i + v_{j-1}^i)k_j^i - u_j^i - v_j^i\big)d\theta \tag{1.32}$$

$$x_j^i = f_j^i(t,X^i) \tag{1.33}$$

$$x_j^i + (\bar{u}_{j-1}^i + \bar{v}_{j-1}^i)k_j^i - \bar{u}_j^i - \bar{v}_j^i = 0,$$

$$j=2,\cdots,J_i,\ i=1,\cdots,I \tag{1.34}$$

$$u = \bar{u} + \tilde{u} \quad v = \bar{v} + \tilde{v} \ (1.18)\cdots(1.21)\ U = u_1 \quad V = v_1 \tag{1.35}$$

$$dz = \beta(t,\theta,z)d\theta + \sqrt{\alpha}\ d\tilde{B}_\theta \tag{1.36}$$

$$h(Y,0,z,U,u) = z - \sum_i e_1^i(Y^i,U^i) - \sum_i\sum_{j=2}^{J_i} e_j^i(0,u_j^i) = H(Y,z,U,u). \tag{1.37}$$

The expectation in (1.30) is relative to the periodic measure, that is, the processes $y_j^i(\theta), z(\theta)$ start with an initial law which keeps the marginal law of these processes periodic of period one.

The equation (1.29) is the dynamic programming equation of the following stochastic control problem called *long term one*:

$$\min_{\bar{U},\bar{V},\bar{u},\bar{v}} E\int_0^1 \nu(t,\bar{X},\bar{Y},\bar{U},\bar{u},\bar{v})\,dt$$

with

$$d\bar{X}_t^i = b_i(t,\bar{X}_t^i)\,dt + \sqrt{a_i}\,dB_t^i$$

$$d\bar{Y}_t^i = L^i(\bar{X}^i - \bar{U}^i - \bar{V}^i)\,dt$$

$$1 \geqslant \bar{U}^i \geqslant 0;\ \bar{V}^i \geqslant 0;\ \bar{X}^i \leqslant \bar{U}^i + \bar{V}^i \quad \text{if } \bar{Y}^i = 1;$$

$$\bar{U}^i + \bar{V}^i \leqslant \bar{X}^i \quad \text{if } \bar{Y}^i = 0$$

$$0 = \bar{x}_j^i + k_j^i(\bar{u}_{j-1}^i + \bar{v}_{j-1}^i) - \bar{u}_j^i - \bar{v}_j^i$$

$$\bar{X}_j^i = f_j^i(t,\bar{X}^i).$$

So the limit stochastic control problem when $\epsilon \to 0$ is decomposed into two stochastic control problems:

1) *one called "short run,"* the purpose of which is to give the optimal allocation during "the week" of the average output of the big seasonal dam and to manage for that purpose the weekly dams

2) *one called "long run,"* the purpose of which is to manage the big seasonal dams knowing the optimal allocation of its output during the "week."

Now we shall give an analytical approximation of the solution of the short run problem.

J. A Relevant Simplified Problem

c being a convex function, we have

$$E\int_0^1 c(h)\,d\theta \geqslant c\left(E\int_0^1 h\,d\theta\right). \tag{1.38}$$

$u_j^i \to e_j^i(y_j^i, u_j^i)$ being concave, c increasing, we get

$$E\int_0^1 c(h)\,d\theta \geqslant c\Bigg(\bar{z} - \sum_1 e_1^i(Y^i,\bar{U}^i) - \sum_i \sum_{j=2}^{J_i} e_j^i(0,\bar{u}_j^i)\Bigg) = c(H(Y,\bar{z},\bar{U},\bar{u})) \tag{1.39}$$

with

$$\bar{z} = E\int_0^1 z\,d\theta. \tag{1.40}$$

The law of z is independent of y and is a solution of

$$\begin{cases} -\dfrac{\partial}{\partial\theta} q_z - D^*(\theta) q_z = 0 \\ q_z \geqslant 0 \\ q_z(0,z) = q_z(1,z) \\ \int q_z(0,z)\,dz = 1 \end{cases} \tag{1.41}$$

D^* being the adjoint of D.

Thus, we get the lower bound $\nu(t,X,Y,\bar{U},\bar{V},\bar{u},\bar{v}) \geqslant c(H(Y,\bar{z},\bar{U},\bar{u}))$ and *the lower bound obtained is reached if we linearize e_j^i around $\bar{u}_j^i$, and if the constraints* (1.18)$\cdots$(1.21) *are not reached* (this is not true in the French case). The following problem (1.42)$\cdots$(1.45) defines the long run stochastic control problem associated with the lower bound (1.39).

$$\begin{cases} \dfrac{\partial W}{\partial t} + \min\limits_{U,u,V,v}\left[\sum\limits_1 A^i(U^i,V^i)W + c(H(Y,\bar{z},U,u))\right] = 0 \\ W(0,X,Y) = W(1,X,Y) + \mu \end{cases} \tag{1.42}$$

with $\bar{z}$ defined by

$$\bar{z}(t) = \int_0^1 \int z q(t,\theta,z)\,d\theta\,dz. \tag{1.43}$$

q is a solution of

$$\begin{cases} -\dfrac{\partial}{\partial\theta} q - \dfrac{\partial}{\partial z}\beta(t,\theta,z)q + \dfrac{1}{2}\alpha\dfrac{\partial^2}{\partial z^2} q = 0, \qquad q \geqslant 0 \\ q(t,\theta,z) = q(t,1,z) \\ \int_0^1 q(t,\theta,z)\,d\theta\,dz = 1 \end{cases} \tag{1.44}$$

and u and v satisfy the constraints

$$x_j^i + k_j^i(u_{j-1}^i + v_{j-1}^i) = u_j^i + v_j^i. \tag{1.45}$$

This simplified problem can be viewed as a problem of managing only the big dams, the stocks of the small ones being equal to zero, with the cost being to meet an average deterministic demand defined by (1.43), (1.44). We have reduced the dimension of the problem to $\dim(X,Y) = 2I \sim 40$. Nevertheless, this problem is intractable. In Section I-K we shall define a suboptimal control problem which is solvable numerically.

K. A Team Problem, The Management of the Big Dams

We shall define a suboptimal problem: the stochastic control problem corresponding to (1.29) in the class of local feedbacks.

The stochastic control problem associated with (1.29) is

$$\min_{U,V,u,v} E\int_0^1 c\Bigg(\bar{z}_t - \sum_i e_1^i(Y_t^i,U_t^1) - \sum_i \sum_{j=2}^{J_i} e_j^i(0,u_{j,t}^i)\Bigg)\,dt \tag{1.46}$$

$$dX_t^i = b_i(t,X_t^i)\,dt + \sqrt{a_i}\,dB_t^i \tag{1.47}$$

$$dY_t^i = L^i(X^i - U^i - V^i)\,dt;\quad 1 \geqslant U^i \geqslant 0;\ V^i \geqslant 0;$$

$$X^i \leqslant U^i + V^i \quad \text{if } Y^i = 1;$$

$$U^i + V^i \leqslant X^i \quad \text{if } Y^i = 0 \tag{1.48}$$

$$0 = x_j^i + k_j^i(u_{j-1}^i + v_{j-1}^i) - u_j^i - v_j^i \tag{1.49}$$

$$x_j^i = f_j^i(t,X^i). \tag{1.50}$$

Let us denote

$$M^i(Y^i,U^i)=\max_{u,v\,(1.49),(1.50)} e_1^i(Y^i,U^i)+\sum_{j=2}^{J_i} e_j^i(0,u_j^i). \tag{1.51}$$

c being an increasing function, (1.46) is equal to

$$\min_{U,V,(1.47),(1.48)} E\int_0^1 c\Big(\bar z_t-\sum_i M^i(Y_t^i,U_t^i)\Big)dt. \tag{1.52}$$

Remark: The problem (1.47)$\cdots$(1.52) is degenerate [no Brownian motion disturbance in (1.48)], so the standard existence theorems cannot be applied [9], [13], [25], [7], [38]. Even the theorem for existence for the degenerate case [21], [22], [35] is not sufficient because of the constraints (1.48). Reference [33] gives the result in the nonperiodic case if we change the constraints

$$X^i\leqslant U^i+V^i,\qquad \text{if } Y^i=1$$
$$X^i\geqslant U^i+V^i,\qquad \text{if } Y^i=0$$

to

$$X^i\geqslant U^i+V^i,\qquad \text{if } Y^i>1$$
$$X^i\geqslant U^i+V^i,\qquad \text{if } Y^i<0$$

(this modification does not change the physical nature of the problem, but ensures us of the existence of a solution).

Let us define the class of local feedbacks:

$\mathfrak{U}_{\text{loc}}=\{U,V: U^i,V^i$ are functions only of the local state t,X^i,Y^i such that there exists an initial law getting the marginal law periodic$\}$.

With the restriction to the class of local feedback controls, the dynamics of the valleys are stochastically independent. That means that the processes (X_t^i,Y_t^i) and $(X_t^{i'},Y_t^{i'})$ have their laws independent for $i\neq i'$.

The probability density of (X_t^i,Y_t^i) is given by

$$\begin{cases} -\dfrac{\partial p^i}{\partial t}-\dfrac{\partial}{\partial X^i}\big[b_ip^i\big]+\dfrac{1}{2}\dfrac{\partial^2}{\partial X^{i2}}\big[a_ip^i\big] \\ \qquad -\dfrac{\partial}{\partial Y^i}\big[L_i(X^i-U^i-V^i)p^i\big]=0 \\ p^i(0,X^i,Y^i)=p^i(1,X^i,Y^i). \end{cases} \tag{1.53}$$

The problem of control can be written

$$\min_{U,V\in\mathfrak{U}_{\text{loc}}}\int_0^1\int\Big(\prod_i p^i\Big)c\Big(\bar z_t-\sum_i M^i\Big)d\Theta\,dt \tag{1.54}$$

with $\Theta=(R\times[0,1])^I$ subject to the constraints (1.53).

We can consider the problem (1.54) as a team problem, each player (the controller of a valley) trying to minimize the same criterion knowing what happens in his valley and the probability laws of the other valleys. The criterion is independent of the starting point.

A sufficient condition for optimality player-by-player (Nash point) is

$$\begin{gathered}\frac{\partial W^i}{\partial t}+\frac{1}{2}a_i\frac{\partial^2}{\partial X^{i2}}+b_i\frac{\partial W^i}{\partial X^i}+\min_{U^iV^i}\Big\{L_i(X^i-U^i-V^i)\frac{\partial W^i}{\partial Y^i} \\ +\int\Big[\prod_{k\neq i}p^k\Big]c\Big(\bar z_t-\sum M^i\Big)d\Theta^i\Big\}=0 \\ \Theta^i=\prod_{k\neq i}(R\times[0,1]) \\ W^i(0,X^i,Y^i)=W^i(1,X^i,Y^i)+\mu^i\end{gathered} \tag{1.55}$$

p_k being the solution of (1.53).

The application of Ito's formula to $W^i(t,X^i,Y^i)$ proves that

$$\begin{aligned}\mu^i&=\min_{U^iV^i}E\int_0^1\int_{\Theta^i}c\Big(\bar z_t-\sum_i M^i\Big)\Big(\prod_{k\neq i}p_k\Big)d\Theta^i\,dt \\ &=\min_{UV}\int_0^1\int_{\Theta}c\Big(\bar z_t-\sum_i M^i\Big)\Big(\prod p_k\Big)d\Theta\,dt=\mu.\end{aligned}$$

So to solve the problem, we can use the relaxation technique proposed in [32], [34].

Step 1: U given.
Step 2: $U^i\to p^i$ $i=1\cdots I$ solving $(1.53)_i$.
Step 3: $i=i+1$ modulo I.
Step 4: $(p^k,U^k\,k\neq i)\to U^i$ solving $(1.55)_i$.
Step 5: $U^i\to p^i$ solving $(1.53)_i$.
Step 6: Go to step 3 until convergence occurs.

It is easy to see that we obtain a decreasing sequence of μ^i, $\mu^i\geqslant 0$, so $\{\mu_i\}$ converges to μ optimal player-by-player cost (Nash point).

L. *Approximation of the Multiple Integrals*

To achieve the numerical solution, we must overcome a last difficulty, the computation of terms like

$$\int_{\Theta^i}\Big(\prod_{k\neq i}p^k\Big)c\Big(\bar z_t-\sum_i M^i\Big)d\Theta^i$$

with $I\sim 20$.

But we can see that we need only the laws of $Q_i=\sum_{k\neq i}M^k$. Using the fact that the M^i are independent random variables, we may apply the central limit theorem to justify the following approximation:

$$Q_i\sim\mathfrak{N}(m_i,\Lambda_i)$$
$$m_i=\sum_{k\neq i}E(M^k)$$
$$\Lambda_i=\sum_{k\neq i}V_r(M^k).$$

The computation of $E(M^k)$ and $V_r(M_k)$ needs only numerical integration in dimension 2. p^k defines completely the law of M^k. So we have

$$\int_{\Theta^i}\left(\prod_{k\neq i} p^k\right)c\left(\bar{z}_t-\sum_1 M^i\right)d\Theta^i$$
$$\cong\int_R \frac{1}{\sqrt{2\Pi\Lambda_i}}\exp\left\{\frac{-(Q-m_i)^2}{2\Lambda_i}\right\}$$
$$\cdot c\left(\bar{z}_t-M^i-Q\right)dQ.$$

M. A Numerical Test of the Algorithm

We solve an example of the team problem described in Section I-K to test the algorithm (in particular, the approximation by the normal law) and to have an idea of the computation time needed.

The discretization of (1.53), (1.55) is explained in Section II, using a Markov chain discretization method [14], [17], [22], [33].

The example treated is the following, with the notations of Section I-L: 8 valleys

$$\bar{z}_t=2/3(1+0.5\sin 2\pi t)$$
$$c(\theta)=\theta^2$$

$$\begin{cases} M_i(Y^i,U^i)=\beta_2^i(1-\exp-\beta_3^iU^i)(1-\beta_4^i\exp-\beta_5^iY^i) \\ \beta_2^i=1/(0.96\times 8) \\ \beta_3^i=-(\log 0.2)/3 \\ \beta_4^i=\beta_5^i=0 \end{cases}$$

$$\begin{cases} X^i(t)=\left(\exp(\xi^i(t)+m^i(t))\right)/6 \\ d\xi^i(t)=-\alpha_1^i\xi^i(t)\,dt+\alpha_2^i dw^i(t) \\ \alpha_1^i=0{,}2 \\ (\alpha_2^i)^2=\alpha_1^i/36 \\ m^i(t)=1+0.2\sin 2\pi t \end{cases}$$

$$\begin{cases} dY^i(t)=L^i(X^i(t)-U^i(t))\,dt \\ L^i=18./e. \end{cases}$$

In this example we have not solved the periodic case ($p^i(0)=p^i(1)$) in (1.53) but have taken a fixed independent initial law.

$p^i(0,\xi^i,Y^i)=q_1(\xi^i)\times q_2(Y^i)$

$q_1=\mathfrak{N}(0,2\times\alpha_2^2/\alpha_1)$ normal law mean 0 and variance $2\alpha_2^2/\alpha_1$

$q_2(Y^i)$ is discretized in 11 points $(0,\cdots,0,0.1,0.4,0.4,0.1)$.

The sequence of cost μ^i of the algorithm is

$$\begin{cases} 192 & 115 & 60 & 26 & 11 & 7 & 6 & 5.8 & \text{first iteration} \\ 5.8 & 5 & 4.6 & 4.4 & 4.2 & 4 & 3.84 & 3.82 & \text{second iteration} \\ 3.66 & 3.57 & 3.48 & 3.39 & 3.29 & 3.2 & 3.16 & 3.16 & \text{third iteration.} \end{cases} \quad (1.56)$$

So to achieve the convergence, three global iterations have been done. In this particular case, it is possible to compute exactly the law of $\Sigma_{k\neq i}M^k$. The difference between this and (1.54) (obtained with the Gaussian approximation) is less than 0.0004.

The computation of an element of the sequence (1.54) costs 3 s of IBM 370-168 for the following discretization:

50 steps in time

11 steps in Y^i

6 steps in X^i.

In view of this computation time, the French case (20 valleys) seems solvable.

II. Numerical Solution of a Local Problem on Real Data

One studies now how to solve a local problem, that is, an approximation of the couple of partial differential equations $(1.53)_i$ and $(1.55)_i$ (in the sequel the index i is dropped).

The functions $b(t,x)$ and $a(t,x)$ are unknown, and they have to be identified before solving $(1.53)_i$ and $(1.55)_i$.

A. Identification of b and a

One has at one's disposal about a 20-year sample on the water supply X_t solution of (1.14). Its probability law P is characterized by the functions b and a.

These functions are approximated by

$$\begin{cases} b(t,x)=\sum_i \theta_i \mathbf{1}_{A_i}(t,x) \\ a(t,x)=\sum_i \mu_i^2\mathbf{1}_{A_i}(t,x) \end{cases} \quad (2.1)$$

where $\{A_i\}$ is a partition of the space time.

Because of the periodicity assumptions made, the sets A_i are chosen as $A_i=\cup_{r=1}^{20}A_{i,r}$ with $A_{i,r+1}=A_{i,r}+1$ (translated).

By Girsanov's formula, if one denotes by Q the probability law of the stochastic process $dy_t=\sqrt{a}\,(t,y_t)dB_t$, one has

$$\frac{dP_t}{dQ_t}=\exp\left\{\int_0^t\frac{b}{a}(s,x_s)\,dx_s-\frac{1}{2}\int_0^t\frac{b^2}{a}(s,x_s)\,ds\right\}. \quad (2.2)$$

It follows that the maximum likelihood estimator $\hat{\theta}$ of θ is given by

$$\hat{\theta}_t^i=\frac{\int_0^t\mathbf{1}_{A_i}(s,X_s)\,dX_s}{\int_0^t\mathbf{1}_{A_i}(s,X_s)\,ds}. \quad (2.3)$$

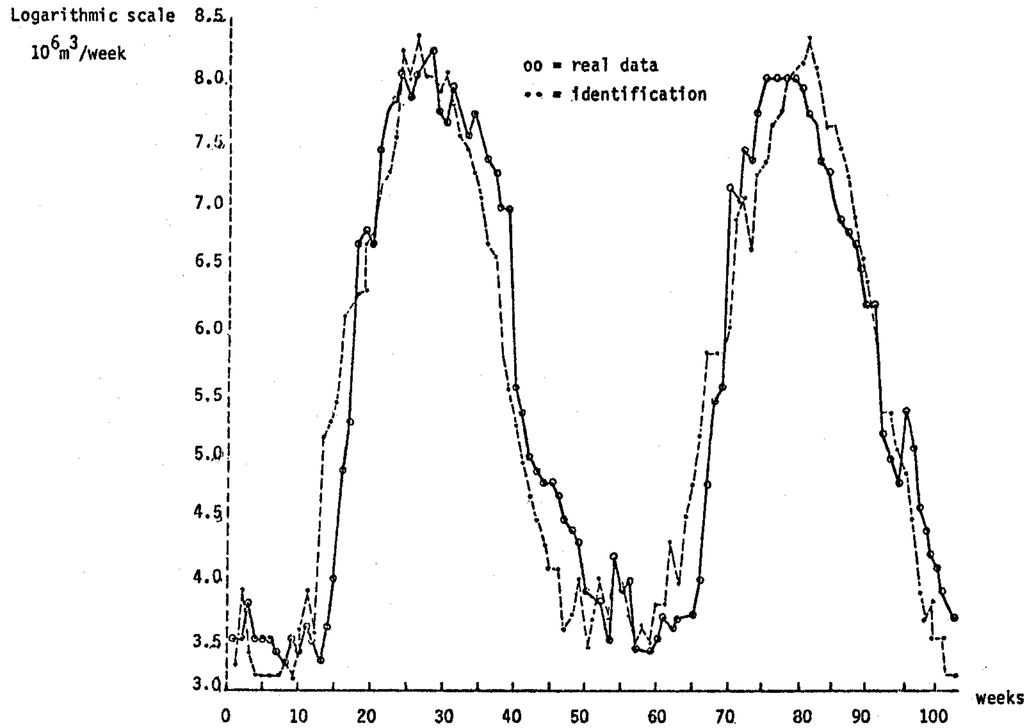


Fig. 1.

Moreover, the quadratic variation of the process X_t on the interval $[0,s]$ is given by

$$\lim_{\substack{\sup_i |t_{i+1}-t_i|\downarrow 0 \\ t_i\in[0,s]}} \sum_i \left(X_{t_{i+1}}-X_{t_i}\right)^2 = \int_0^s a(s,X_s)\,ds \text{ a.e.} \tag{2.4}$$

It follows that

$$\mu_k^2 = \frac{\lim_{\sup|t_{i+1}-t_i|\downarrow 0} \sum_{i\in I_k} \left(X_{t_{i+1}}-X_{t_i}\right)^2}{\int_0^s \mathbf{1}_{A_k}(t,X_t)\,dt} \tag{2.5}$$

where the summation is taken over $I_k=\{i;(t_i,X_{t_i})\in A_k\}$ and s such that $\int_0^s \mathbf{1}_{A_k}(t,X_t)\,dt>0$.

In the discrete time case, (2.3) and (2.5) are approximated by

$$\begin{cases} \hat{\theta}_k = \dfrac{1}{T_k}\sum_{i\in I_k}\left(X_{t_{i+1}}-X_{ti}\right) \\ \hat{\mu}_k^2 = \dfrac{1}{T_k}\sum_{i\in I_k}\left(X_{t_{i+1}}-X_{t_i}-\hat{\theta}_k(t_{i+1}-t_i)\right)^2, \\ \quad\text{with } T_k=\sum_{i\in I_k}(t_{i+1}-t_i). \end{cases} \tag{2.6}$$

The statistical properties of the estimators are studied in [14], [18] and [19]. In particular, their variances can be estimated by $V_r(\hat{\theta}_k)\approx(\hat{\mu}_k^2/T_k)$ and $V_r(\hat{\mu}_k^2)=(2\hat{\mu}_k^4/|I_k|)$.

Fig. 1 shows the numerical results obtained with real data (dam of TIGNES in the French Alps); it presents a part of the observed trajectory (two years) and a trajectory obtained by simulation after identification.

Remark: For computational convenience (for the control problem), the identified process is $X_t=\rho(t,\xi(t))$ where ρ is a regular, strictly increasing $[0,1]$ valued function and ξ_t is the real input. The $\rho(t,\cdot)$ function chosen is the distribution function of a normal law with mean $\bar{m}(t)$ and variance $\bar{r}^2(t)$ where $\bar{m}(t)$ and $\bar{r}^2(t)$ are the sample mean and variance; this transformation resets the diffusion X_t in the bounded set $[0,1]$ with unreachable boundaries. One has

$$\xi(t)=\exp\left\{\bar{\mu}(t)+\bar{r}(t)N^{-1}(X(t))\right\},$$
$$N(v)=\int_{-\infty}^{v}\frac{1}{\sqrt{2\Pi}}\exp-\frac{x^2}{2}\,dx. \tag{2.7}$$

B. Numerical Computation of a Closed-Loop Control

In this part one solves numerically a local problem, that is, approximately step 4 of the algorithm described before. By the stochastic interpretation of the discretized problem, one gets, furthermore, a method of resolution of step 2 of the algorithm.

The stochastic control problem for a valley can be formulated (in the finite horizon case):

$$\max E\int_0^T g(t,y_t,u_t)\,dt+H(y_T) \tag{2.8}$$

$$dX=b(t,X_t)\,dt+\sqrt{a}\,(t,X_t)\,dB_t \tag{2.9}$$

$$dY_t=\begin{cases} -(\xi-u)^-\,dt, & \text{if } Y_t=1 \\ (x\text{-}u)\,dt, & \text{if } 0<Y_t<1 \\ (\xi-u)^+\,dt, & \text{if } Y_t=0 \end{cases} \tag{2.10}$$

with the notations of remark of Section II-A.

g is a gain function (value of the power produced by the dam) and H is a final cost which gives us the value of the water at the end of the period of management.

Let us denote by $V(s,x,y)$ the Bellman function, that is,

$$V(s,x,y)=\max_u E\left\{\int_s^T g(t,Y_t,u_t)\,dt + H(Y_T)|X(s)=x,Y(s)=y\right\}. \quad (2.11)$$

V is the solution of the following dynamic programming equation:

$$\frac{\partial V}{\partial t}+b\frac{\partial V}{\partial x}+\frac{a}{2}\frac{\partial^2 V}{\partial x^2} + \max_u\left[(\xi-u)\frac{\partial V}{\partial y}+g(t,y,u)\right]=0, \qquad \begin{matrix}0<x<1\\0<y<1\end{matrix} \quad (2.12)$$

with $a(t,0)=a(t,1)=0$, $(\xi-u)$ replaced by $(\xi-u)^+$[resp. $-(\xi-u)^-$] if $y=0$ [resp. $y=1$] and $V(T,x,y)=H(y)$.

The problem of existence, unicity, and regularity of a solution of (2.12) necessary to have the probabilistic interpretation of V (2.11) is not studied here. One can, however, give a numerical solution to (2.12) by a method of discretization given in [22], [33] which allows us to interpret the discretized problem as a Markov chain control problem. Equation (2.12) is only written here to obtain a discretized problem which admits a solution thanks to its probabilistic interpretation, although (2.12) is a degenerate partial differential equation [33].

Let $(\Delta t,\Delta x,\Delta y)$ by the discretization steps. Equation (2.12) is approximated by the following equations where (t,x,y) take the discretized values

$$t\in\{T,T-\Delta t,\cdots,0\}$$
$$x\in\{0,\Delta x,\cdots,N_x\cdot\Delta x=1\}=G_x$$
$$y\in\{0,\Delta y,\cdots,N_y\cdot\Delta y=1\}=G_y$$

$$\frac{\partial V}{\partial t}(t,x,y)\leftrightarrow\frac{V(t+\Delta t,x,y)-V(t,x,y)}{\Delta t}$$

$$b(t,x)\frac{\partial V}{\partial x}\leftrightarrow\left[b^+(x)\frac{V(x+\Delta x)-V(x)}{\Delta x} - b^-(x)\frac{V(x)-V(x-\Delta x)}{\Delta x}\right](t+\Delta t,y)$$

$$(\xi-u)\frac{\partial V}{\partial y}\leftrightarrow\left[(\xi-u)^+\frac{V(y+\Delta y)-V(y)}{\Delta y} - (\xi-u)^-\frac{V(y)-V(y-\Delta y)}{\Delta y}\right](t+\Delta t,x)$$

$$\frac{a}{2}(t,x)\frac{\partial^2 V}{\partial x^2}\leftrightarrow\left[\frac{a}{2}\frac{V(x+\Delta x)+V(x-\Delta x)-2V(x)}{(\Delta x)^2}\right]\cdot(t+\Delta t,y).$$

Then one can solve (2.12) in an explicit way. Indeed, this approximation is valid at the boundary because $b^+=0$ if $x=1$ and $b^-=0$ if $x=0$.

Now we set

$$\begin{cases} p_t^u(x,y|x,y)=1-\dfrac{|b(t,x)|\Delta t}{\Delta x} - \dfrac{|(\xi-u)|\Delta t}{\Delta x}-\dfrac{a(t,x)\Delta t}{\Delta x} \\ p_t^u(x+\Delta x,y|x,y)=\dfrac{b^+(t,x)\Delta t}{\Delta x}+\dfrac{a(t,x)\Delta t}{2(\Delta x)^2} \\ p_t^u(x-\Delta x,y|x,y)=\dfrac{b^-(t,x)\Delta t}{\Delta x}+\dfrac{a(t,x)\Delta t}{2(\Delta x)^2} \\ p_t^u(x,y+\Delta y|x,y)=\dfrac{(\xi-u)^+\Delta t}{\Delta y} \\ p_t^u(x,y-\Delta y|x,y)=\dfrac{(\xi-u)^-\Delta t}{\Delta y}. \end{cases} \quad (2.13)$$

If $x=1$ (res. $x=0$), one has $b^+=0$ (res. $b^-=0$) and $a=0$, and so $p_t^u(x+\Delta x,y|x,y)=0$(res. $p_t^u(x-\Delta x|x,y)=0$).

For $y=1$ (res. $y=0$), one sets $(\xi-u)^+=0$ [res. $(\xi-u)^-=0$] in (2.13).

One remarks that for Δt sufficiently small, one has $p_t^u\geqslant 0$ and $\Sigma_{x,y}p_t^u(x,y|x_0,y_0)=1$, and thus one can interpret the matrix $p_t^u(\cdot|\cdot)$ as the transition matrix of a Markov chain $\{\eta_t=(x_t,y_t)\}$, $t=0,\cdots,T$.

$$P(\eta_{t+1}=(x',y')|\eta_t=(x,y))=p_t^u(x',y'|x,y). \quad (2.14)$$

Now the discretized equation (2.12) can be written as

$$\begin{aligned} V(t,x,y)=\max_u\{&V(t+\Delta t,x,y)\cdot p_t^u(x,y|x,y)\\ &+V(t+\Delta t,x+\Delta x,y)\cdot p_t^u(x+\Delta x,y|x,y)\\ &+V(t+\Delta t,x-\Delta x,y)\cdot p_t^u(x-\Delta x,y|x,y)\\ &+V(t+\Delta t,x,y+\Delta y)\cdot p_t^u(x,y+\Delta y|x,y)\\ &+V(t+\Delta t,x,y-\Delta y)\cdot p_t^u(x,y-\Delta y|x,y)\\ &+g(t,y,u)\cdot\Delta t\} \qquad (t=T-1,\cdots,0) \quad (2.15)\end{aligned}$$

with $V(T,x,y)=H(y)$.

Equation (2.15) is the discrete Bellman equation of the discrete time control problem $\max_u V^u(0,x_0,y_0)$ with

$$V^u(t,x,y)=\mathop{E}_{x,y}\left\{\sum_{\tau=t}^{T-1} g(\tau,y_\tau,u_\tau(x_\tau,y_\tau))\Delta t+H(y_T)\right\} \quad (2.16)$$

where $\eta_t=(x_t,y_t)$ is a Markov chain with space state $G_x\times G_y$, transition matrix (2.13), and initial condition (x_0,y_0).

Because of the probabilistic interpretation (2.16), one can calculate directly the transition probabilities

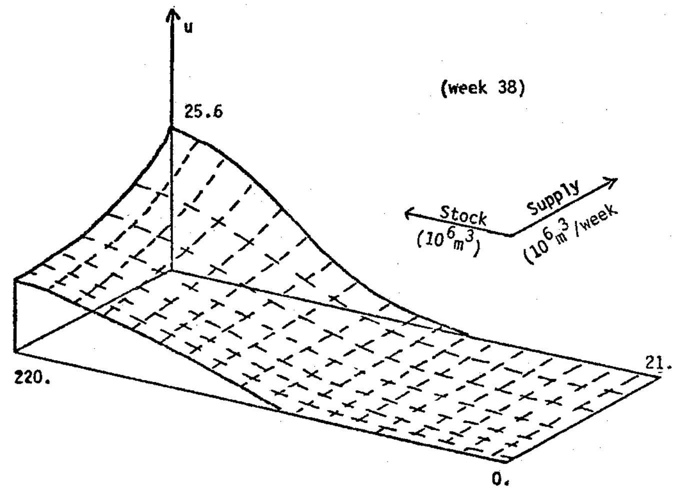


Fig. 2.

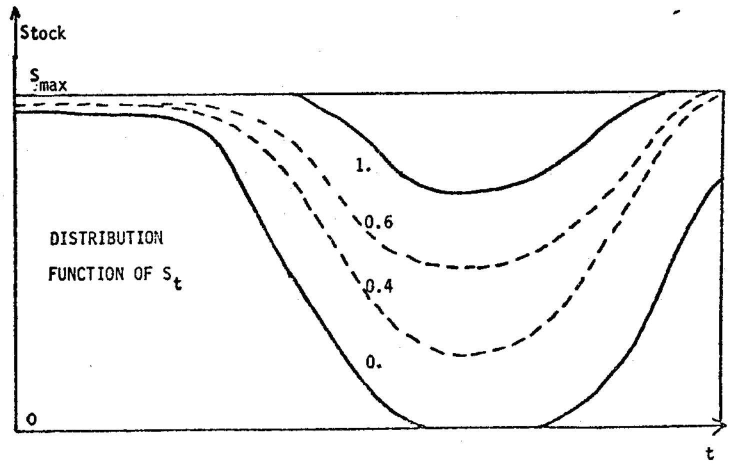


Fig. 3.

$$P^u(x_r=x',y_r=y'|x_t=x,y_t=y) = p^u_{t,r}(x',y'|x,y), \qquad r=t+1,\cdots,T$$

with the following formulas:

$$p^u_{t,t}(x',y'|x,y)=\delta(x,y;x',y') \quad \text{(Kronecker symbol)}$$

$$p^u_{t,t+k+1}(x'y'|x,y) = \sum_{x_1=0}^{N_x}\sum_{y_1=0}^{N_y} p^u_{t,t+k}(x_1,y_1|x,y)\times p^u_{t+k}(x',y'|x_1,y_1)$$

(discrete time Fokker–Planck equation). (2.17)

One gets directly an approximation of (1.53) necessary for step 2 of the algorithm. The cost function (2.16) admits the representation

$$V^u(t,x,y)=\sum_{k=0}^{T-t-1}\left\{\sum_{x_1=0}^{N_x}\sum_{y_1=0}^{N_y} g(t+k,y_1,u_{t+k}(x_1,y_1)) \cdot p^u_{t,t+k}(x_1,y_1|x,y)\Delta t\right\} + \sum_{x_1}\sum_{y_1} p^u_{t,T}(x_1,y_1|x,y)H(y_1).$$

C. Numerical Results

The maximization problem (2.15) can be solved in an explicit way using an optimality necessary condition which is the discretization of the necessary condition

$$g'_u(t,y,u)=\frac{\partial V}{\partial y}(t,x,y). \qquad (2.18)$$

The left-hand side of (2.18) is the (deterministic) gain obtained from a "marginal" unit of water turbined at time t with the stock y, and the right-hand side of (2.18) represents the mathematical expectation of the future (stochastic) gain starting in state (x,y). It is clear that for $0<u<1$, (2.18) must be verified.

Fig. 2 shows the control obtained, that is, the quantity of water to be turbined as a function of the water supply x and the stock y at time t fixed (here week number 38). The computation has been made with real data (dam of TIGNES).

Fig. 3 represents the time evolution of the isoprobability curves for the stock, obtained from (2.13) and (2.17).

One sees that the optimal policy makes the stock of water decrease to a low level for a few months.

In [14] another example is treated with another dam, and one gets a policy which keeps the level of water near its upper boundary (in that last case, the functions g depends strongly on y).

Appendix I
A Periodic Dynamic Programming Equation

Let $u(t,x)$ be a feedback, and a controlled diffusion

$$dx_t=b(t,x_t,u(t,x_t))\,dt+\sigma(t,x_t)\,dB_t \qquad (I.1)$$

where $t\to b(t,x,u),\sigma(t,x)$ are periodic functions of period 1.

Let us denote $Q^u(s,x,t,y)dy$ as the conditional probability law of x_t knowing that $x_s=x$, and $\mathcal{U}=\{u:R^+\times R\to R$ such that $t\to u(t,x)$ is periodic of period 1 and there exists a unique p_s verifying

$$p_s(y)=\int Q^u(s,x,s+1,y)p_s(x)\,dx\Big\}. \qquad (I.2)$$

Let $c(t,x,u)$ be the cost function, periodic in t. We can define the stochastic control problem:

$$\min_{u\in\mathcal{U}}\lim_T\frac{1}{T}\int_0^T c(t,x_t,u(t,x_t))\,dt = \min_{u\in\mathcal{U}} E\int_0^1 c(t,x_t,u(t,x_t))\,dt, \qquad (I.3)$$

E meaning the expectation relative to this "periodic law."

Let us consider the partial differential equation

$$\frac{\partial W}{\partial t}+\frac{1}{2}a\frac{\partial^2}{\partial x^2}W+\min_u\left[b(t,x,u)\frac{\partial W}{\partial x}+c(t,x,u)\right]=0$$
$$W(0,x)=W(1,x)+\mu. \qquad (I.4)$$

Let us suppose that there exists a unique $(\mu, W)\, W\in C^{1,2}, \mu\in R$ solving (I.4) (that is the main difficulty) then μ is the optimal cost (I.3). Indeed, let u be a feedback belonging to $\mathcal{U}$; applying Ito's formula to the $W(t,x)$ solution of (I.4), we have

$$E[W(1,x_1)-W(0,x_0)]=E\int_0^1\left(\frac{\partial W}{\partial t}+\frac{\partial W}{\partial x}b(t,x_t,u(t,x_t))+\frac{1}{2}a(t,x_t)\frac{\partial^2}{\partial x^2}W\right)dt. \qquad (I.5)$$

The left-hand side of (I.5)$=\int_0^1(W(1,x)-W(0,x))p_0(x)\,dx$ [by (I.4)] $=-\mu$; the right-hand side of (I.5) $=-E\int_0^1 c(t,x_t,u(t,x_t))\,dt$, so we have

$$\mu\leqslant E\int_0^1 c(t,x_t,u(t,x_t))\,dt \qquad \forall u\in\mathcal{U}$$

and the equality is reached for $u'(t,x)\in\arg\min[b(t,x,u)(\partial W/\partial x)+c(t,x,u)]$. So we have

$$\mu=E\int_0^1 c(t,x_t,u^*(t,x_t))\,dt$$
$$\leqslant E\int_0^1 c(t,x_t,u(t,x_t))\,dt \qquad \forall u\in\mathcal{U}.$$

Remark: Some periodic parabolic equations are studied in [28]. In [23] and [29] the ergodic stochastic control problem is studied in the time homogeneous case.

Appendix II
An Averaging and Singular Perturbation Problem

A. The Averaging Theorem

Let us consider the following control problem:

$$dx_t=b_1(t,t/\epsilon,x_t,y_t,u_t)\,dt+\sigma_1\,dB_t^1$$
$$dy_t=1/\epsilon\, b_2(t,t/\epsilon,x_t,y_t,u_t)\,dt+1/\sqrt{\epsilon}\,\sigma_2\,dB_t^2$$

$\frac{1}{2}\sigma_i^2=a_i \quad i=1,2;\ \ B_t^i$ independent Brownian motion

$\theta\to b_i(t,\theta,x,y,u)$ periodic of period 1

$c(t,\theta,x,y,u)$ cost function one periodic in θ.

$$\min_u E\int_0^T c(t,t/\epsilon,x_t,y_t,u_t)\,dt. \qquad (II.1)$$

Let us define

$$W^\epsilon(t,x,y)=\min_u E\int_t^T c\,ds.$$

W^ϵ is the solution of the following Bellman equation:

$$\begin{cases}\dfrac{\partial W^\epsilon}{\partial t}+a_1\dfrac{\partial^2 W^\epsilon}{\partial x^2}+1/\epsilon a_2\dfrac{\partial^2 W^\epsilon}{\partial y^2}\\ \quad+\min_u\Big[b_1(t,t/\epsilon,x,y,u)\dfrac{\partial W^\epsilon}{\partial x}\\ \quad+\dfrac{b_2}{\epsilon}(t,t/\epsilon,x,y,u)\dfrac{\partial W^\epsilon}{\partial y}\\ \quad+c(t,t/\epsilon,x,y,u)\Big]=0\\ W^\epsilon(T,x,y)=0.\end{cases} \qquad (II.2)$$

The problem is to study the asymptotic behavior of W^ϵ when $\epsilon\to 0$.

We conjecture the following theorem.

Theorem: If (t,x) *is considered as a parameter,* $\forall u(t,\theta,x,y)$

$$\begin{cases}-\dfrac{\partial\tilde q}{\partial\theta}-\dfrac{\partial}{\partial y}[b_2\circ u\tilde q]+\dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}[a_2,\tilde q]=0 \quad \tilde q\geqslant 0\\ \tilde q(t,0,x,y)=\tilde q(t,1,x,y) \qquad \text{with } b_2\circ u(t,\theta,x,y)\\ \quad =b_2(t,\theta,x,y,u(t,\theta,x,y))\\ \int\tilde q(t,0,x,y)\,dy=1\end{cases} \qquad (II.3)$$

has a weak solution (measure), then the W^ϵ *solution of* (II.2) *converges at least pointwise to* $\overline{W}$, *the solution of*

$$\begin{cases}\dfrac{\partial\overline{W}}{\partial t}+\dfrac{1}{2}a_1\dfrac{\partial^2\overline{W}}{\partial x^2}+\mu\left(t,x,\dfrac{\partial\overline{W}}{\partial x}\right)=0\\ \overline{W}(T,x)=0\end{cases} \qquad (II.4)$$

where $\mu(t,x,\lambda)$ *is solution of*

$$\begin{cases}\dfrac{\partial\tilde W}{\partial\theta}+\dfrac{1}{2}a_2\dfrac{\partial^2\tilde W}{\partial y^2}+\min_u\Big[b_2(t,\theta,x,y,u)\dfrac{\partial\tilde W}{\partial y}\\ \quad+\lambda b_1(t,\theta,x,y,u)+c(t,\theta,x,y,u)\Big]=0\\ \tilde W(t,0,x,y)=\tilde W(t,1,x,y)+\mu(t,x,\lambda).\end{cases} \qquad (II.5)$$

Moreover, if

$$\int\tilde q(t,\theta,x,y)|y|\,dy<\infty,$$

then

$$\int_0^1\int b_2(t,\theta,x,y,u(t,\theta,x,y))\tilde q(t,\theta,x,y)\,d\theta\,dy=0. \qquad (II.6)$$

Remarks: At present we have a proof only in the particular case $b_2=0$; $a_2=0$ [15].

A proof in the following deterministic case, $a_1, a_2=0$; b_1 and b_2 linear in x,y,u independent of θ, c quadratic in $x,y,$

is given in [11], [31]. In this case, $\tilde{q}(t,x,y)=\delta_{y^*(t,x)}$ where $y^*(t,x)$ is the solution of $b_2(t,x,y)=0$.

In [1]–[6] other averaging theorems are given.

Singular perturbation techniques applied to the control of partial differential equations are given in [26], [27].

Let us give the outlines of a formal proof. For that purpose, let $p(t,\theta,x,y)$ be the probability measure of the state θ_t,x_t,y_t where $\theta_t=t/\epsilon$ modulo 1.

p is the solution of the following Fokker–Planck equation:

$$\begin{cases} -\dfrac{\partial p}{\partial t}+\dfrac{1}{2}a_1\dfrac{\partial^2}{\partial x^2}p-\dfrac{\partial}{\partial x}[b_1\circ up] \\ \quad +1/\epsilon\left\{-\dfrac{\partial p}{\partial \theta}+\dfrac{1}{2}a_2\dfrac{\partial^2 p}{\partial y^2}-\dfrac{\partial}{\partial y}[b_2\circ up]\right\}=0 \\ p(0,\theta,x,y)=\delta(0,x_0,y_0) \\ p(t,0,x,y)=p(t,1,x,y). \end{cases} \tag{II.7}$$

The stochastic control problem can be viewed as the control of the partial differential equation (II.7), the cost function being

$$W^\epsilon(0,x_0,y_0)=\min_u\int_0^T\int_0^1\int c(t,\theta,x,y,u(t,\theta,x,y))\cdot p(t,\theta,x,y)\,dt\,d\theta\,dx\,dy.$$

Let us denote by $\tilde{q}(t,\theta,x,y)$ the solution of

$$\begin{cases} -\dfrac{\partial \tilde{q}}{\partial \theta}+\dfrac{1}{2}a_2\dfrac{\partial^2\tilde{q}}{\partial y^2}-\dfrac{\partial}{\partial y}[b_2\circ u\tilde{q}]=0, \qquad \tilde{q}\geq 0 \\ \tilde{q}(t,0,x,y)=\tilde{q}(t,1,x,y) \\ \int\tilde{q}(t,0,x,y)\,dy=1 \end{cases} \tag{II.8}$$

and by $\bar{q}(t,x)$ the solution of

$$\begin{cases} -\dfrac{\partial \bar{q}}{\partial t}+\dfrac{1}{2}a_1\dfrac{\partial \bar{q}}{\partial x^2}-\dfrac{\partial}{\partial x}\left\{\left[\int b_1\circ u\tilde{q}\,d\theta\,dy\right]\bar{q}\right\}=0 \\ \bar{q}(0,x)=\delta_{x_0}. \end{cases} \tag{II.9}$$

A solution of (II.7) can be written

$$p^\epsilon(t,\theta,x,y)=\bar{q}(t,x)\tilde{q}(t,\theta,x,y)+\epsilon r^\epsilon(t,\theta,x,y)$$

with r^ϵ the density of a bounded (independent of ϵ) measure.

Assuming $|c(t,\theta,x,y)|$ bounded, we can prove by the maximum theorem [8] that

$$W^\epsilon(0,x_0,y_0)\underset{\epsilon\to 0}{\to}W(0,x_0)=\min_u\int_0^T\int_0^1\int\bar{q}(t,x)\tilde{q}(t,\theta,x,y)\cdot c(t,\theta,x,y,u(t,\theta,x,y))\,dt\,d\theta\,dx\,dy.$$

Now we have to solve the following control problem:

$$W(0,x_0)=\min_u\int\bar{q}(t,x)\tilde{q}(t,\theta,x,y)\cdot c(t,\theta,x,y,u(t,\theta,x,y))\,dt\,d\theta\,dx\,dy$$

$$\text{subject to the constraints (II.8), (II.9).} \tag{II.10}$$

A condition of optimality for this problem is given by (II.4), (II.5).

Multiplying (II.8) by y and integrating by part, we obtain

$$\int_0^1\int_{-m}^m b_2\tilde{q}\,d\theta\,dy=\int_0^1\left[b_2\tilde{q}y-\frac{1}{2}\frac{\partial}{\partial y}\cdot[a_2\tilde{q}]y+\tfrac{1}{2}\tilde{q}a_2\right]_{-m}^m d\theta. \tag{II.11}$$

The condition $\int\tilde{q}|y|\,dy<\infty$ gives (II.6) by passing to the limit in (II.11) $m\to\infty$.

By this method, we can prove that $\overline{W}(0,x)\leq J(u)\forall u$ Lipschitz, under very general assumptions, where $J(u)$ is $\lim_{\epsilon\to 0}$ of the cost for the feedback u.

On regularity conditions of the solution of (II.5), in the stationary case, [37] gives a complete proof.

B. Interpretation of the Averaged Problem

Equations (II.4) and (II.5) can be viewed as two coupled stochastic control problems for which the dynamic programming equations are

$$\begin{cases} \dfrac{\partial\overline{W}}{\partial t}+\dfrac{1}{2}a_1\dfrac{\partial^2\overline{W}}{\partial x^2}+\min_v\left[\dfrac{v\partial\overline{W}}{\partial x}+\max_\zeta \nu(t,x,v,\zeta)\right]=0 \\ \overline{W}(T,x)=0 \end{cases} \tag{II.4$'$}$$

$$\begin{cases} \dfrac{\partial\tilde{W}}{\partial\theta}+\dfrac{1}{2}a_2\dfrac{\partial^2\tilde{W}}{\partial y^2}+\min_u\Big[b_2(t,\theta,x,y,u)\dfrac{\partial\tilde{W}}{\partial y} \\ \quad +c(t,\theta,x,y,u)+\zeta(b_1(t,\theta,x,y,u)-v)\Big]=0 \\ \tilde{W}(t,0,x,y)=\tilde{W}(t,1,x,y)+\nu(t,x,v,\zeta). \end{cases} \tag{II.5$'$}$$

This problem is obtained when we write (II.10):

$$\min_v\min_u\int\bar{q}\tilde{q}c\,dt\,d\theta\,dx\,dy.$$

$$\int b_1\tilde{q}\,d\theta\,dy=v$$

ζ is the dual variable associated with the constraint $\int b_1\tilde{q}\,d\theta\,dy=v$. The usefulness of the formulation (II.4′) and (II.5′) is that in some case we can obtain analytically a good approximation of $\max_\zeta \nu(t,x,v,\zeta)$.

Equation (4′) will be called *the long run problem*, and (5′) *the short run* one.

So the limit problem is decomposed into two stochastic control problems. The first is the *short run one*, knowing t and x,

$$\begin{cases} dy_\theta = b_2(t,\theta,x,y_\theta,u_\theta)\,d\theta + \sigma_2\,dB_\theta^2 \\ E\int_0^1 b_1(t,\theta,x,y_\theta,u_\theta)\,d\theta = v \\ \min\limits_u E\int_0^1 c(t,\theta,x,y_\theta,u_\theta)\,d\theta = \nu(t,x,v) \end{cases}$$

where the expectation is relative to a probability law for the initial condition, such that the marginal law of the process y_θ is periodic. θ is the short run time.

The second is the *long run* one:

$$\begin{cases} dx_t = v\,dt + \sigma_1\,dB_t^1 \\ \min E\int_0^T \nu(t,x_t,v_t)\,dt \end{cases}$$

where t is the long run time.

So the short run control problem can be viewed as an optimal allocation of the resource v given by the long run problem.

Remark: The short run problem defined here is different from the "fast system" defined in [11]; it is the coupled system (II.4′) (II.5′) which is the "slow system" of [11].

Conclusion

In this paper we have studied a particular large-scale system. The techniques used here have much more general interest: reduction of the dimension of the system (for example, by singular perturbation techniques), and reduction of the class of the admissible strategies (for example, class of feedbacks decoupling the dynamic of the system).

This method seems implementable for the management of a large system of dams with a reasonable computation cost.

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