Second Order Theory of Min-Linear Systems and its Application to Discrete Event Systems

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Abstract

A Second Order Theory is developed for linear systems over the $(\min,+)$-algebra; in particular the classical notion of correlation is extended to this algebraic structure. It turns out that if we model timed event graphs as linear systems in this algebra, this new notion of correlation can be used to study stocks and sojourn times, and thus to characterize internal stability (boundedness of stocks and sojourn times). This theory relies heavily on the algebraic notion of residuation which is briefly presented.

1 Introduction

In [4, 6, 5], a linear system theory analogous to the conventional theory has been developed for a particular class of Discrete Event Dynamic Systems (DEDS) called Timed Event Graphs (TEG). This theory extends the notions of state space, impulse response and transfer function to TEG's. The periodic behavior of these systems has been characterized and a spectral theory has been developed. The key feature which allows extending all these classical concepts is a general notion, called $(\min,+)$-linearity. A system is $(\min,+)$-linear if the min of the inputs produces the min of the corresponding outputs, and if, when a constant is added to *all* the inputs, then the same constant is added to all the outputs. TEG's are particular discrete $(\min,+)$-linear systems, but the theory is also suited for continuous time systems [10]. In this general framework, the input-output relation of time-invariant systems can be expressed as an inf-convolution of the inputs with the impulse response, and the theory looks very similar to the conventional one. However, all these results belong to the "first order linear theory", i.e. the quantities which are considered are linear functions of the inputs. This is not the case for other interesting quantities such as stocks and sojourn times. The stock in a place depends on the difference between inputs and outputs of this place. Similarly, the sojourn time is the difference between input and output times. It should be realized that, in the $(\min,+)$-algebra, such differences are nonlinear functions. However we will show that stocks and sojourn times are particular cases of quasi-bilinear quantities called "correlations" by analogy with the second order theory of stochastic processes. The study of correlations forms a "second order" theory which will appear to be the adequate tool for dealing with stability issues of DEDS. In §2, we summarize the first order $(\min,+)$-linear theory. In §3, we introduce a few elements of residuation theory needed to deal with correlations. Then, in §4, the second order theory is applied to TEG's.

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2 First Order Theory of $(\min,+)$ Linear Systems

In this section, we briefly recall how TEG's can be modeled by linear equations over some particular dioids. A more detailed account can be found in [6, 10].

2.1 $(\min,+)$ Equations

We consider the TEG of Figure 1. For each transition of the graph, say for the one labeled X_1, $x_1(t)$ denotes the number of firings up to time t (*counter* function). The initial marking (tokens represented by dots) and

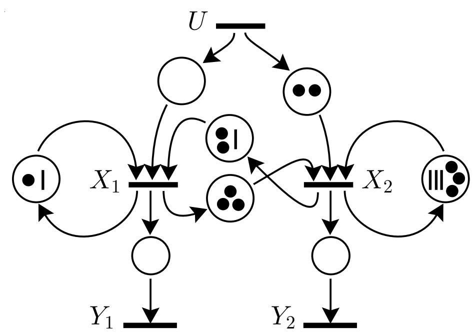

Figure 1: A simple timed event graph

the holding times in the places in time units (number of bars) are given in Figure 1. Then the following inequalities are obtained:

$$\begin{aligned} &x_1(t) \le \min(1 + x_1(t-1), 2 + x_2(t-1), u(t)) \ , \\ &x_2(t) \le \min(3 + x_1(t), 3 + x_2(t-3), 2 + u(t)) \ , \\ &y_1(t) \le x_1(t) \ , \ \ y_2(t) \le x_2(t) \ . \end{aligned} \tag{1}$$

Since we are interested in transition firings occurring at the earliest possible time, the maximal solution (which achieves equalities in (1)) is the one of interest. We shall give several linear representations of this system.

2.2 Linear Recurrent Representation in $\overline{\mathbb{R}}_{\min}$

Let $\overline{\mathbb{R}}_{\min}$ denote the set $\mathbb{R} \cup \{\pm\infty\}$ endowed with min as addition, denoted $\oplus$, and $+$ as multiplication, denoted $\otimes$ (e.g. $2 \otimes 3 \oplus 7$ stands for $\min(2+3, 7) = 5$ with the usual priority rules). We conventionally set $(-\infty) \otimes (+\infty) = +\infty$. The sign $\otimes$ will be omitted as usual when this causes no risk of confusion. This algebraic structure is a particular example of a *dioid* [6], i.e. there exists a *zero* element $\varepsilon = +\infty$, a *unit* element $e = 0$, $\oplus$ is idempotent ($a \oplus a = a$), the usual combinatorial rules (associativity, distributivity) hold, and the zero element ε is absorbing for multiplication ($x \otimes \varepsilon = \varepsilon$). With this notation, System (1) can be rewritten as

$$\begin{array}{rcl} x_1(t) & \leq & 1x_1(t-1) \oplus 2x_2(t-1) \oplus u(t) \ , \\ x_2(t) & \leq & 3x_1(t) \oplus 3x_2(t-3) \oplus 2u(t) \ , \\ y_1(t) & \leq & x_1(t) \ , \ \ y_2(t) \leq x_2(t) \ , \end{array} \tag{2}$$

in which linearity becomes more apparent.

Remark 2.1 Another important dioid is $\overline{\mathbb{R}}_{\max}$, the dual dioid of $\overline{\mathbb{R}}_{\min}$ (min is replaced by max, and the convention $(-\infty) \otimes (+\infty) = -\infty$ is adopted). In this dioid, $\overline{\mathbb{R}}_{\max}$-linear *dater* equations can be written [6]. ∎

2.3 Representation in the Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

Let us introduce the shift in counting, γ, and the shift in dating, δ, defined respectively by:

$$[\gamma(x)](t) = 1 + x(t), \qquad [\delta(x)](t) = x(t-1) \ .$$

The operators γ and δ commute and are $\overline{\mathbb{R}}_{\min}$-linear. The powers of γ and δ obey the following important absorption rules [6]:

$$\begin{array}{rl} (i) & \gamma^{\nu} \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')} \ , \\ (ii) & \delta^{\tau} \oplus \delta^{\tau'} = \delta^{\max(\tau,\tau')} \ . \end{array} \tag{3}$$

The set of operators in TEG's can be represented by the *dioid* of formal series with two commuting variables, γ and δ, integer exponents and boolean coefficients, together with the simplification rules (3). For reasons which will become clear later, we allow negative exponents as well. The construction of this dioid is done more explicitly in [6]. Because of the simplification rules, this dioid is called $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ (pronounced "min max γ δ"). The unit element will again be denoted e, and the zero, ε. System (2) can now be expressed as follows:

$$\begin{array}{l} x \leq \begin{pmatrix} \gamma\delta & \gamma^2\delta \\ \gamma^3 & \gamma^3\delta^3 \end{pmatrix} x \oplus \begin{pmatrix} e \\ \gamma^2 \end{pmatrix} u = Ax \oplus Bu \ , \\ y \leq \begin{pmatrix} e \\ e \end{pmatrix} x = Cx \ . \end{array} \tag{4}$$

Introducing the *rational* operation *: $A^* = \bigoplus_{i=0}^{+\infty} A^i$, one easily checks that the maximal solution of (4) is given by

$$y = CA^*Bu \stackrel{def}{=} Hu \ . \tag{5}$$

H is called the transfer matrix of System (4). Let E be the identity matrix. The matrix A^*, which is the least solution of $X = AX \oplus E$ (after the order defined in §3.1), can be computed using Gaussian elimination [1, 9, 6], provided that the * of scalars can be computed. For instance, after some computations using the simplification rules (3), we obtain:

$$A^* = \begin{pmatrix} (\gamma\delta)^* & \gamma^2\delta\,(\gamma\delta)^* \\ \gamma^3\,(\gamma\delta)^* & \left(\gamma^3\delta^3\right)^* \end{pmatrix} \ ,$$

$$H = A^*B = \begin{pmatrix} (\gamma\delta)^* \\ \gamma^2\left(e \oplus \gamma^2\delta\right)\left(\gamma^3\delta^3\right)^* \end{pmatrix} \ . \tag{6}$$

In order to interpret the transfer matrix H, we recall that the counter function canonically associated with a series s is the unique nondecreasing function $\mathcal{C}_s : \mathbb{Z} \to \overline{\mathbb{Z}}$ such that $s = \bigoplus_{t\in\mathbb{Z}} \gamma^{\mathcal{C}_s(t)}\delta^t$ (with the convention $\gamma^{+\infty} = \varepsilon$, $\gamma^{-\infty} = (\gamma^{-1})^*$). Similarly, the dater function $\mathcal{D}_s$ is the unique nondecreasing function $\mathbb{Z} \to \overline{\mathbb{Z}}$ such that $s = \bigoplus_{n\in\mathbb{Z}} \gamma^n\delta^{\mathcal{D}_s(n)}$ (with $\delta^{+\infty} = \delta^*$, $\delta^{-\infty} = \varepsilon$). Since one possible representation[1] of e is $\bigoplus_{t\geq 0} \gamma^t\delta^0$, the dater function associated with e is

$$\mathcal{D}_e(t) = \begin{cases} -\infty & \text{if } t < 0 \ ; \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

This essentially means that an infinite number of tokens (numbered $0, 1, \ldots$) have been placed at the input at time 0; therefore, e plays the role of the *impulse* and $He = H$ is the *impulse response* of the system. For instance, since the first entry of the transfer matrix is equal to $H_1 = (\gamma\delta)^* = e \oplus \gamma\delta \oplus \gamma^2\delta^2 \oplus \cdots$, we obtain $\mathcal{D}_{H_1}(0) = 0, \mathcal{D}_{H_1}(1) = 1, \ldots$ which means that after time 0, the transition X_1 is fired once every time unit.

2.4 Representation by Inf-Convolutions

We introduce the dioid $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$ of functions $\mathbb{R} \to \overline{\mathbb{R}}$, endowed with the pointwise min as addition and the inf-convolution as multiplication, i.e.

$$(u \otimes v)(t) = \bigoplus_{\tau\in\mathbb{R}} u(t-\tau)v(\tau)$$

(the product in the right-hand side denotes the product in $\overline{\mathbb{R}}_{\min}$, and the notation $\bigoplus$ for inf is extended to infinite families, see §3). For instance, $y_1 = H_1(u)$ rewrites

$$\begin{array}{rcl} y_1(t) & = & [(\gamma\delta)^*(u)](t) \\ & = & \inf\{u(t), 1 + u(t-1), 2 + u(t-2), \ldots\} \end{array}$$

which is nothing but the inf-convolution of u with

$$h(t) = \begin{cases} t & \text{if } t \in \mathbb{N} \ ; \\ \varepsilon & \text{otherwise.} \end{cases}$$

More generally, it is shown in [10] that a general SISO time-invariant $\overline{\mathbb{R}}_{\min}$-linear system $\mathcal{S}$ can be represented by the input-output relation

$$[\mathcal{S}(u)](t) = \bigoplus_{\tau\in\mathbb{R}} h(t-\tau) \otimes u(\tau) \ . \tag{8}$$

We go from representation (5) to representation (8) using the map $s \mapsto \mathcal{C}_s$ defined in §2.3. More specifically, we extend $\mathcal{C}_s$ to noninteger values by taking $\mathcal{C}_s$ constant on intervals $[n, n+1)$, with $n \in \mathbb{Z}$. Then, $\mathcal{C}$ becomes a dioid morphism from $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ to $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$, which maps (5) to (8). The representation (8) obviously extends to the MIMO case: h becomes a matrix with entries in $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$. Dual *sup-convolution* representations can also be written for dater functions.

[1]Due to the simplification rules, there are several formal equivalent representations of the same element in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$—see [6] for details.

3 Algebraic Theory of Correlations

3.1 Residuation in Complete Dioids

Residuation is a general notion in lattice theory which allows defining "pseudo-inverses" of some isotone maps. Here, we are basically concerned with solving equations of the type $ax = b$, i.e. in "inverting", in a sense to be defined later, the map $x \mapsto ax$. Therefore, we shall recall here the basic facts about residuation of multiplicative maps in complete dioids, and refer the interested reader to [3] for an extensive account of residuation theory.

Natural order relation in dioids In any dioid $\mathfrak{D}$, the *natural* order relation is defined as follows:

$$a \preceq b \Leftrightarrow a \oplus b = b \ . \tag{9}$$

The least upper bound of a and b, $a \vee b$, is equal to $a \oplus b$. Because $\varepsilon \oplus b = b$, ε is the least element in $\mathfrak{D}$. We say that the dioid $\mathfrak{D}$ is *complete* if

(i) all (possibly infinite) families $\{x_i\}_{i \in I}$ in $\mathfrak{D}$ admit a least upper bound, denoted for obvious reasons $\bigoplus_{i \in I} x_i$, and

(ii) the product is distributive with respect to the least upper bound, that is:

$$a \left(\bigoplus_{i \in I} x_i \right) = \bigoplus_{i \in I} a x_i, \quad \left(\bigoplus_{i \in I} x_i \right) a = \bigoplus_{i \in I} x_i a \ . \tag{10}$$

In complete dioids, the greatest lower bound $\wedge$ always exists and is characterized by:

$$a \wedge b = \bigoplus_{\substack{x \preceq a \\ x \preceq b}} x \ .$$

Generalized quotients in complete dioids

Definition 3.1 *In a complete dioid, the left and right quotients of a and b are defined respectively by:*

$$\begin{aligned} a \circ\!\!\backslash b &= \max \{x \mid ax \leq b\} \ , \\ a \circ\!\!/ b &= \max \{x \mid xb \leq a\} \end{aligned} \tag{11}$$

*(*max *emphasizes that the least upper bound itself belongs to the subset).*

Indeed, since ε is absorbing, the set $\{x \mid ax \leq b\}$ is nonempty and it is stable under $\oplus$ because of (10), thus left and right quotients are well defined. In other words, we have weakened the notion of solution of $ax = b$ to that of subsolution and called left quotient the maximal subsolution. The basic properties of quotients are:

$$x \mapsto a \circ\!\!\backslash x \ \text{ is isotone,} \tag{12}$$

$$x \mapsto x \circ\!\!\backslash b \ \text{ is antitone} \tag{13}$$

(f is isotone, resp. antitone, if $a \preceq b \Rightarrow f(a) \preceq f(b)$, resp. $f(a) \succeq f(b)$),

$$b \text{ invertible} \Rightarrow b \circ\!\!\backslash a = b^{-1} a \ , \tag{14}$$

$$a \circ\!\!\backslash a \succeq e \ , \tag{15}$$

$$(ab) \circ\!\!\backslash c = b \circ\!\!\backslash (a \circ\!\!\backslash c) \ , \tag{16}$$

$$c \circ\!\!\backslash (ab) \succeq (c \circ\!\!\backslash a) b \ , \tag{17}$$

$$b \text{ invertible} \Rightarrow c \circ\!\!\backslash (ab) = (c \circ\!\!\backslash a) b \ , \tag{18}$$

$$(a \oplus b) \circ\!\!\backslash c = (a \circ\!\!\backslash c) \wedge (b \circ\!\!\backslash c) \ , \tag{19}$$

$$c \circ\!\!\backslash (a \wedge b) = (c \circ\!\!\backslash a) \wedge (c \circ\!\!\backslash b) \ . \tag{20}$$

Of course, all the properties already mentioned for left quotients admit dual formulations for right quotients. For instance, (16) becomes:

$$c \circ\!\!/ (ab) = (c \circ\!\!/ b) \circ\!\!/ a \ .$$

Matrix residuation Next we generalize Definition 3.1 to matrices with entries in a complete dioid. The greatest matrix $X \in \mathfrak{D}^{p \times k}$ such that $AX \leq B$ with $A \in \mathfrak{D}^{n \times p}$ and $B \in \mathfrak{D}^{n \times k}$ will again be denoted $A \circ\!\!\backslash B$ (a similar definition is adopted for right matrix quotient). Simple formulae relate matrix quotients to scalar quotients [2]:

$$(A \circ\!\!\backslash B)_{ij} = \bigwedge_{l=1}^{n} A_{li} \circ\!\!\backslash B_{lj} \ , \tag{21}$$

$$(A \circ\!\!/ B)_{ij} = \bigwedge_{l=1}^{k} A_{il} \circ\!\!/ B_{jl} \ . \tag{22}$$

Let us define the *dual trace* of a square matrix A by $Tr^{\wedge}(A) = \bigwedge_i A_{ii}$. For all n-dimensional vectors u, v, we obtain the counterpart of the classical identity $Tr(uv^T) = v^T u$:

$$Tr^{\wedge}(u \circ\!\!/ v) = v \circ\!\!\backslash u \tag{23}$$

($v \circ\!\!\backslash u$ is a scalar, and $u \circ\!\!/ v$ is an $n \times n$ matrix). It is worth noting that in general $a \circ\!\!\backslash b$ and $a \circ\!\!/ b$ have the same dimensions as $a^T b$ and ab^T respectively (see formulae (21), (22)). Let us we give a few examples of quotients in familiar dioids.

Residuation of product in $\overline{\mathbb{R}}_{\min}$ We immediately check that:

$a \circ\!\!\backslash b = b - a$	if a and b are finite,
$a \circ\!\!\backslash (-\infty) = (-\infty)$	for all a,
$a \circ\!\!\backslash \varepsilon = \varepsilon$	for all a finite,
$\varepsilon \circ\!\!\backslash a = (-\infty)$	for all a,
$(-\infty) \circ\!\!\backslash a = \varepsilon$	if $a \neq (-\infty)$.

It should be noticed that $\varepsilon \otimes (-\infty) = +\infty - \infty = \varepsilon = +\infty$ whereas $\varepsilon \circ\!\!/ \varepsilon = +\infty - \infty = -\infty$, which shows that the notation $a - b$ is ambiguous for infinite values of a and b.

Residuation of product in $\overline{\mathbb{R}}_{\max}$ The only changes by comparison with $\overline{\mathbb{R}}_{\min}$ are

$a \circ\!\!\backslash (+\infty) = (+\infty)$	for all a,
$\varepsilon \circ\!\!\backslash a = (+\infty)$	for all a,
$(+\infty) \circ\!\!\backslash a = \varepsilon$	if $a \neq (+\infty)$.

Residuation of inf-convolution product There is an important formula which relates the scalar quotient in $\overline{\mathbb{R}}_{\min}$ with the quotient in $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$, namely:

$$(u \circ\!\!\!\backslash y)(t) = \bigwedge_{\tau\in\mathbb{R}} [u(\tau - t) \circ\!\!\!\backslash y(\tau)] \ , \tag{24}$$

where the former quotient $\circ\!\!\!\backslash$ denotes the residuation of the inf-convolution product and the latter denotes the residuation in $\overline{\mathbb{R}}_{\min}$ already introduced. In a perhaps more suggestive way, we can write for *finite* u and y:

$$(u \circ\!\!\!\backslash y)(t) = \sup_{\tau\in\mathbb{R}} [y(\tau) - u(\tau - t)] \ . \tag{25}$$

3.2 Correlations

Definition 3.2 *Let u, v be two vectors with respective dimensions n and p. The matrix $u \circ\!\!\!/ v$ is called the* correlation matrix *of u over v. If $u = v$, it is called the* autocorrelation matrix *of u.*

The relation with standard correlations will be discussed in §4.1. Here we just establish the algebraic properties of these quantities. In particular, we investigate how correlations evolve through linear systems. The following result is noticeable.

Theorem 3.3 (Increasing Correlation Principle) *Let $y = Hu$ and $z = Hv$ be the outputs corresponding to inputs u and v. Then*

$$y \circ\!\!\!/ z \succeq (v \circ\!\!\!\backslash u)(H \circ\!\!\!/ H) \ , \tag{26}$$

$$y \circ\!\!\!\backslash z \succeq (u \circ\!\!\!\backslash v) Tr^{\wedge}(H \circ\!\!\!/ H) \ . \tag{27}$$

Proof We have

$$\begin{aligned}
y \circ\!\!\!/ z &= (Hu) \circ\!\!\!/ (Hv) \\
&= [(Hu) \circ\!\!\!/ v] \circ\!\!\!/ H && \text{(by (16))} \\
&\succeq [H(u \circ\!\!\!/ v)] \circ\!\!\!/ H && \text{(by (17))} \\
&\succeq [H Tr^{\wedge}(u \circ\!\!\!/ v) E] \circ\!\!\!/ H \\
&\succeq [H(v \circ\!\!\!\backslash u) E] \circ\!\!\!/ H && \text{(by (23))} \\
&\succeq [(v \circ\!\!\!\backslash u) H] \circ\!\!\!/ H && (v \circ\!\!\!\backslash u \text{ is scalar}) \\
&\succeq (v \circ\!\!\!\backslash u)(H \circ\!\!\!/ H) && \text{(using (17) again).}
\end{aligned}$$

Formula (27) is obtained by similar calculations. ■

Since $H \circ\!\!\!/ H \succeq E$, (27) implies that

$$y \circ\!\!\!\backslash z \succeq u \circ\!\!\!\backslash v \ , \tag{28}$$

which means that in the scalar case, the correlation of output signals is never less than the correlation of inputs. In the case of autocorrelations, (26) becomes:

$$y \circ\!\!\!/ y \succeq (u \circ\!\!\!\backslash u)(H \circ\!\!\!/ H) \ . \tag{29}$$

Because $(u \circ\!\!\!\backslash u) \succeq e$, (29) yields a second correlation principle, which states that the autocorrelation of the output is at least equal to the intrinsic correlation $H \circ\!\!\!/ H$ of the system.

Theorem 3.3 suggests the importance of quotients of the form $A \circ\!\!\!/ A$. We give a simple algebraic characterization of these quotients when A is a square matrix without proof.

Proposition 3.4 *The following statements are equivalent:*

(i) there exists a matrix B such that $A = B \circ\!\!\!/ B$;

(ii) there exists a matrix B such that $A = B^*$;

(iii) $A = A \circ\!\!\!/ A$;

(iv) $A = A^*$;

(v) $A^2 = A$ and $A \succeq E$ ($x \mapsto Ax$ is a closure mapping [3]).

3.3 Rationality and Residuation

We now consider the problem of effectively computing quotients, particularly for power series in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. Of course, the power series which arise in the study of TEG's are not arbitrary, but they are *rational* (i.e. they are obtained by a finite combination of sums, products, and * of polynomials). From formulae (19) and (20), we note that quotients and $\wedge$ operations are closely related. For this reason, we first study the $\wedge$ of rational series.

Theorem 3.5 *The $\wedge$ of two rational series in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is also rational.*

Sketch of proof This result can be derived from a theorem of Eilenberg and Schützenberzer ([8], Theorem III), which states that the intersection of rational sets in a commutative monoid is rational. In our case, a simpler (but less general) argument exists for *causal* rational series. In [6], it is shown that rational causal series can be written as

$$s = P \oplus \gamma^n \delta^t Q (\gamma^\nu \delta^\tau)^* \tag{30}$$

where P and Q are causal polynomials such that $\deg P \leq (n-1, t-1)$ and $\deg Q \leq (\nu - 1, \tau - 1)$. Therefore, it is sufficient to show that the $\wedge$ of series of type (30) is again of the form (30). This essentially results from the distributivity of $\wedge$ with respect to $\oplus$, and the fact that

$$\gamma^n \delta^t (\gamma^\nu \delta^\tau)^* \wedge \gamma^{\overline{n}} \delta^{\overline{t}} (\gamma^{\overline{\nu}} \delta^{\overline{\tau}})^* = \widetilde{P} \oplus \gamma^{\widetilde{n}} \delta^{\widetilde{t}} \widetilde{Q} (\gamma^{\widetilde{\nu}} \delta^{\widetilde{\tau}})^*$$

where $\widetilde{\tau}/\widetilde{\nu} = \min(\tau/\nu, \overline{\tau}/\overline{\nu})$. ■

Theorem 3.6 *The quotient of two rational series in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is also rational.*

Instead of proving this theorem in the general case, we will concretely compute quotients of some rational series in Example 4.7 and show that they are rational.

4 Second Order Theory

4.1 Correlations and Stocks

We now consider the problem of computing the stocks in timed event graphs. Let u, v be two counter functions associated with two transitions immediately before and after a place p and let $\overline{S}_{u,v}(t)$ denote the stock of tokens in place p at time t. We have the fundamental relation:

$$\overline{S}_{uv}(t) = \overline{S}_{uv}(0) + u(t) - v(t) \ . \tag{31}$$

Introducing the variation of stock after time 0:

$$S_{uv}(t) = \overline{S}_{uv}(t) - \overline{S}_{uv}(0) \ , \tag{32}$$

(31) can be written as $S_{uv}(t) = u(t) - v(t)$. More generally, for two transitions u and v connected by a path, $S_{uv}(t)$ represents the variation of the total number of tokens from time 0 to time t of all the places along the path. If there are several parallel paths from u to v, this variation is clearly the same for all these paths.

Definition 4.1 (Stock matrix) *Let u, resp. v, be an n-dimensional, resp. p-dimensional vector the entries of which are counter functions. The stock matrix $S_{uv}(t)$ at time t from u to v is defined by $S_{uv}(t) = u(t) \not\circ v(t)$.*

When $u_i(t)$ and $v_j(t)$ are finite, we have:

$$(S_{uv}(t))_{ij} = u_i(t) - v_j(t) \ , \tag{33}$$

so that S_{uv} generalizes the definition of stocks to infinite values. Another interesting quantity is $S^+_{uv}(t) = u(t) - v(t^-)$. S^+ is different from S when v is not left-continuous at time t.

Given two vectors of dater functions u and v, we dually define the sojourn time matrix $T_{uv}(n)$ for the event numbered n by:

$$T_{uv}(n) = v(n) \not\circ u(n) \tag{34}$$

(for dater functions, the quotient is of course that of $\overline{\mathbb{R}}_{\max}$). The inversion of u and v between (33) and (34) is necessary in order to obtain a nonnegative sojourn time for the place or the path between the input transition u_i and the output transition v_j. The main result of this section is an immediate consequence of (24).

Theorem 4.2 (Stock evaluation formula) *Let u, v be vectors the entries of which are counter functions. For all $t \in \mathbb{R}$, we have*

$$S_{uv}(t) \leq (u \not\circ v)(0) \ . \tag{35}$$

Moreover, $(u \not\circ v)(0)$ is the tightest constant bound. Also $S^+_{uv}(t) \leq (u \not\circ v)(0^+)$.

Since $S_{uv}(t) = -S_{vu}(t)$ for all finite $u(t)$ and $v(t)$, (35) may also serve to get a lower bound for S_{uv}. This formula admits a counterpart in $\mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$. Let u, v be two vectors with entries in $\mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$ and $\mathcal{C}_u$, $\mathcal{C}_v$ be the associated counter function vectors. We have:

$$S_{\mathcal{C}_u \mathcal{C}_v}(t) \leq [\mathcal{C}(u \not\circ v)](0), \tag{36}$$

where $\not\circ$ denotes the quotient in $\mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$. Sojourn times can also be bounded by way of the inequality:

$$[\mathcal{D}(v \not\circ u)^T](0) \leq T_{\mathcal{D}_u \mathcal{D}_v}(n) \ . \tag{37}$$

We let the reader find the dual of (35) for sojourn times using the quotient associated with sup-convolution of daters.

Remark 4.3 In writing the stock evaluation formula, we have used the usual order relation $\leq$ which is just the opposite of $\preceq$ in the dioid $\overline{\mathbb{R}}_{\min}$. In the following, we will always use $\leq$ when the concrete meaning of the results is concerned. It is important to note that u and v are strongly correlated (in the practical sense that the backlog between u and v remains small) when $(u \not\circ v)(0)$ is large with respect to $\preceq$. ∎

Remark 4.4 When $x = u = v$, the *autocorrelation* simply measures the maximal variation of x in a window of width t:

$$(x \not\circ x)(t) = \bigwedge_{\tau \in \mathbb{R}} [x(\tau) \not\circ x(\tau - t)] \ , \tag{38}$$

i.e. $\sup_\tau [x(\tau) - x(\tau - t)]$ for finite x. Formula (23) with $u = v = x$ gives the autocorrelation of a vector of time functions. We have:

$$x \backslash\!\!\circ\, x = Tr^\wedge(x \not\circ x) = \bigwedge_i x_i \not\circ x_i \ , \tag{39}$$

i.e. $x \backslash\!\!\circ\, x$ is equal to the min of the autocorrelations of all the entries of x. Therefore, the "scalar product" $x \backslash\!\!\circ\, x$ can be interpreted as a measure of the time-space dispersion of x. ∎

Remark 4.5 With Formula (24), the analogy with the conventional correlation $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(\tau - t) y(\tau) \, d\tau$ should be clear. ∎

4.2 Some Applications

The properties of correlations can be used to algebraically derive a well known property of Event Graphs.

Proposition 4.6 *The number of tokens in any circuit of a timed event graph is constant.*

Proof The variation of the number of tokens in a circuit is equal to the storage variation S_{xx} from any transition x of the circuit to itself. Then, the stock evaluation formula can be rewritten (for a finite $x(t)$) as $-(x \not\circ x)(0) \leq S_{xx}(t) \leq (x \not\circ x)(0)$. But $x \not\circ x \succeq e$ implies that $(x \not\circ x)(0) \leq e(0) = 0$, and therefore $0 \leq S_{xx}(t) \leq 0$. This shows that the number of tokens remains constant. ∎

Example 4.7 For the event graph of Figure 1, applying Proposition 3.4 and with (6), we get: $(H \not\circ H)_{11} = (\gamma\delta)^* \not\circ (\gamma\delta)^* = (\gamma\delta)^*$. For $(H \not\circ H)_{12}$, (20) yields:

$$\begin{aligned} &(\gamma\delta)^* \not\circ (\gamma^2(e \oplus \gamma^2\delta)(\gamma^3\delta^3)^*) = \\ &\qquad \gamma^{-2}(\gamma\delta)^* \not\circ (\gamma^3\delta^3)^* \wedge \gamma^{-4}\delta^{-1}(\gamma\delta)^* \not\circ (\gamma^3\delta^3)^* \ . \end{aligned} \tag{40}$$

But from the monotonicity properties (12) and (13), we get $(\gamma\delta)^* \succeq (\gamma\delta)^* \not\circ (\gamma^3\delta^3)^* \succeq (\gamma\delta)^* \not\circ (\gamma\delta)^* = (\gamma\delta)^*$. Moreover, $\gamma^{-4}\delta^{-1}(\gamma\delta)^* \succeq \gamma^{-4}\delta^{-1}\gamma\delta(\gamma\delta)^* = \gamma^{-3}(\gamma\delta)^* \succeq \gamma^{-2}(\gamma\delta)^*$, which shows that the second term in the right-hand side (40) is greater than the first term. Thus, $H_{12} = \gamma^{-2}(\gamma\delta)^*$. The other correlations can be computed in a similar way. We obtain:

$$\begin{aligned} (H \not\circ H)_{21} &= \gamma^2\delta^{-1}(\gamma\delta)^* \ , \\ (H \not\circ H)_{22} &= \left(e \oplus \gamma^2\delta\right)\left(\gamma^3\delta^3\right)^* \ , \end{aligned}$$

which in terms of counting functions yields:

$$\begin{aligned} -\mathcal{C}_{(\gamma\delta)^*}(0) = 0 \leq S_{X_1X_1} \leq 0 = \mathcal{C}_{(\gamma\delta)^*}(0) \ , \\ -\mathcal{C}_{(H \not\circ H)_{21}}(0) = -3 \leq S_{X_1X_2} \leq -2 = \mathcal{C}_{(H \not\circ H)_{12}}(0) \ , \\ -\mathcal{C}_{(H \not\circ H)_{22}}(0) = 0 \leq S_{X_2X_2} \leq 0 = \mathcal{C}_{(H \not\circ H)_{22}}(0) \ . \end{aligned}$$

From the bounds on $S_{X_1X_1}$ and $S_{X_2X_2}$, we check again that the number of tokens in a circuit is constant. In

Figure 2, we have drawn the counter functions associated with H_1 (black points) and H_2 (white points). The difference between the two functions at a given time gives the stock. It takes the values -2 and -3, which is consistent with the bounds given here above. More trivial bounds can also be directly derived: the variation of stock in place $X_1 \to X_2$ is greater than -3 (otherwise, the number of tokens in this place would be negative). Moreover, it cannot exceed 2 (because the number of tokens in a circuit is constant, the number of tokens in the other place of the circuit, viz. $X_2 \to X_1$, would be negative). Thus, we only have a priori that $-3 \leq S_{X_1 X_2} \leq 2$ (and it is not obvious that these bounds are tight). ■

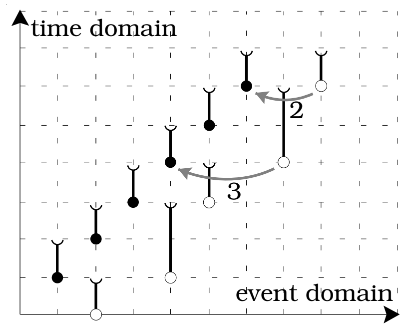

Figure 2: Counter functions and stocks

4.3 Stability

Definition 4.8 (Internal stability) *A timed event graph is internally stable if for all inputs, the internal stocks remain bounded.*

Given a representation in $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$ of $\mathcal{S}$ defined by $x = Ax \oplus Bu$, the internal stocks are described by the matrix $S_{xx}(t) = x(t) {/\!\!\!\circ}\, x(t)$ (cf. (33)). By the stock evaluation formula, we see that $\mathcal{S}$ is stable iff for all input u, we have:

$$\forall i, j, \quad (x {/\!\!\!\circ}\, x)(0)_{ij} < +\infty \ . \qquad (41)$$

We have the following characterization, for which we skip the proof.

Theorem 4.9 (Condition of Internal Stability) *$\mathcal{S}$ is stable iff $[(A^* B) {/\!\!\!\circ}\, (A^* B)](0) < +\infty$.*

Stability has something to do with the "structure" of matrices A, B, C. Recall that a graph G can be associated with $A\ (n \times n), B\ (n \times p), C\ (q \times n)$ in the following way: G is made up of n nodes labeled x_i, p nodes u_k, q nodes y_j, there is an arc from x_i to x_l if $A_{li} \neq \varepsilon$, there is an arc from u_k to x_i if $B_{ik} \neq \varepsilon$, an arc from x_i to y_j if $C_{j,i} \neq \varepsilon$. $\mathcal{S}$ is structurally controllable if for all nodes x_i of G, there is a path from some u_k to x_i. It is structurally observable if for all x_i, there is a path of x_i to some y_j.

Proposition 4.10 (Feedback stabilization) *A structurally controllable and observable causal system can be stabilized by feedback.*

Sketch of proof We have to find a matrix K such that $\mathcal{S}' : \ x = A'x \oplus Bu; \ \ y = Cx$ with $A' = A \oplus BKC$ is stable. From the increasing correlation principle, we get $x {/\!\!\!\circ}\, x \succeq [(Bu) \circ\!\!\!\backslash (Bu)][(A')^* {/\!\!\!\circ}\, (A')^*] \succeq (A')^* {/\!\!\!\circ}\, (A')^*$ (this is simply Formula (29) in which Bu replaces u). Hence, with Proposition 3.4, we have $x {/\!\!\!\circ}\, x \succeq (A')^*$, so that it is sufficient to find K such that all the entries of $(A')^*(0)$ be finite. A sufficient condition is that the graph associated with A' be strongly connected and that each circuit contain at least one token. This is possible thanks to the connectivity assumption. ■

Example 4.11 The TEG represented in Figure 3 is not stable. For instance, if tokens arrive in U_1 at the rate of 2 tokens per unit of time, then tokens accumulate without bound in the place from X_1 to X_2 since the throughput of X_2 is limited to one token per unit of time. However, the system is structurally controllable and observable and it can be stabilized by the feedback K shown in Figure 3. ■

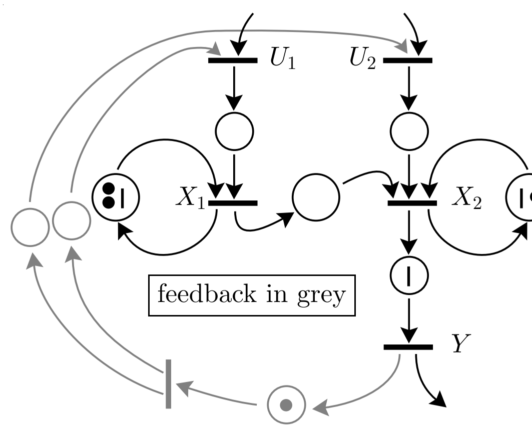

Figure 3: An unstable timed event graph with a stabilizing feedback

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