CONVEX ANALYSIS AND SPECTRAL ANALYSIS OF TIMED EVENT GRAPHS

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Abstract

Using the algebra of dioids, we further examine the analogy between timed event graphs and conventional linear systems by showing that some periodic inputs of the former behave as cosine inputs for the latter. In particular, we give a meaning to such notions as "phase shift" and "amplification gain", which allow us to talk about the Black and Bode plots for discrete event systems. In this theory, classical concepts of Convex Analysis such as inf-convolution and Fenchel conjugate play the parts that convolution and Laplace transform play in the conventional case.

1 Introduction

Event graphs constitute a special class of Petri nets in which transitions admit several incoming and outgoing arcs whereas places admit single upstream and downstream arcs. In this way, only synchronization constraints (logical AND) can be represented, whereas alternative choices (corresponding to logical OR) cannot be represented. A *fork*—several outgoing arcs from a transition—represents for example the simultaneous broadcasting of messages in several directions. A *join*—several incoming arcs—represents the requirement of simultaneous availability of resources in order to perform a task. Considering *timed* event graphs and performance evaluation, a fork mathematically translates into the equality of either the dates of the corresponding event occurrences or the numbers of event occurrences up to any given time. As for joins, they involve max or min operations whether we consider dates or number of events.

These max or min operations would *a priori* classify timed event graphs as nonlinear and even nonsmooth systems. But it has been shown that, using the so-called algebra of dioids, it is possible to consider these systems as linear systems, and to develop a theory for them which bears much resemblance with the conventional linear system theory. A fairly comprehensive account of this emerging theory has been given in [3]. It has been shown how to develop "state space" models in either the time domain (playing with numbers of event occurrences at time t called *counters* c_t) or the event domain (using dates of occurences of events of rank n called *daters* d_n). These state space models have their "input-output" model counterparts (transfer matrices). A two-dimensional event-time domain representation has also been proposed. Among other results and concepts, it has been shown that the asymptotic behaviour of such autonomous systems can be characterized in terms of eigenvalues and eigenvectors; stability and stabilization by dynamic output feedback have been investigated, etc...

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The aim of the present paper is to pursue this striking analogy with the conventional theory by studying the counterpart of the Black and the Bode plots. In particular, it is well known that, after a transient behaviour, a cosine input through a stable time-invariant linear system yields an amplified and phase shifted cosine output at the same frequency. Mathematically, this may be stated as follows: eigenfunctions of rational transfer functions are cosines at various frequencies and the eigenvalues are the complex numbers the modulus and the argument of which represent respectively the amplification gains and the phase shifts. The variation of these quantities as a function of the frequency are represented by the Black or the Bode plots.

We shall obtain similar results with some adaptation for the class of timed event graphs. The role of cosines will be played by some periodic inputs; it will be shown that any linear system driven by this kind of input delivers the same kind of output up to some phase shift and amplification gain. The role of phase shift and amplification gain are interchanged when switching from the counter to the dater point of view.

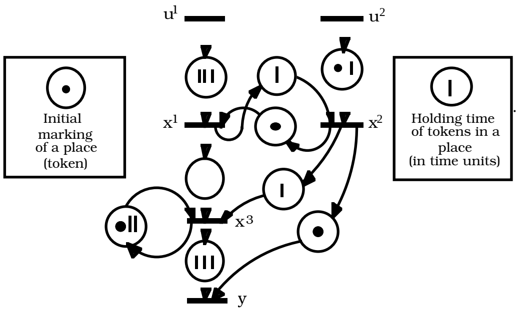
In the classical case, the amplification gain and the phase shift at each frequency ω are obtained by evaluating the Laplace transform of the transfer function, say $H(s)$, at $s = j\omega$. It should be realized that in this process, $H(s)$ which is initially considered as a *formal* rational function of the derivative operator—or more precisely of its Laplace counterpart s—is also used as a *numerical* function. The same kind of trick will be used in the case of transfer functions of timed event graphs. It will be shown that considering the numerical function associated with a transfer function amounts to using the Fenchel transform of the impulse response. Here comes the connection with Convex Analysis which will be further elaborated hereafter.

The results presented in this paper can be explained using the time domain counter representation, the event domain dater representation or the two-dimensional domain representation. We shall limit ourselves to the first and third points of view after having given the relationships that allow one to pass from one to another representation. Using for example the counter description, a striking analogy will be established between conventional linear and min-linear systems, convolution and inf-convolution, and Laplace and Fenchel transforms. Most of the results will be stated here without or with sketchy proofs because of the lack of space. A more complete account of this theory will appear elsewhere.

2 Event graph models using the dioid algebra

2.1 Dater approach

In this section, we recall some basic facts about modeling event graphs with dioids and some properties of the algebra. The reader is referred to [3] for details. An example of a timed event graph is given below with its initial marking and with holding times put on places (with no loss of generality, we may assume that transitions are fired instantaneously, *i.e.* with null duration—see [3]). For every transition, say for one labelled x^i, we number the successive firings in the order they occur using an index $n \in \mathbb{Z}$ starting from an arbitrary, possibly negative, initial value (common to all transitions). There is an absolute clock giving a beep every unit of time, these beeps being also numbered in $\mathbb{Z}$ by an index t. When the firing of x_i numbered n occurs while the clock delivers the beep numbered t, we set $x^i_n = t$. The mapping $n \mapsto x^i_n$ associated with transition x^i is called the *dater function* of this transition.



It is not hard to see that the following equations must hold for the above example

$$\begin{array}{ll} x^1_n = \max\left(x^2_{n-1}, u^1_n + 3\right) & x^2_n = \max\left(x^1_n + 1, u^2_{n-1} + 1\right) \\ x^3_n = \max\left(x^1_n, x^2_n + 1, x^3_{n-1} + 2\right) & y_n = \max\left(x^2_{n-1}, x^3_n + 3\right) \end{array} \quad (1)$$

The dioid $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$ is an agebraic structure with two operations, max as addition (denoted by $\oplus$) and $+$ as multiplication (denoted by $\otimes$ or simply omitted), where $\varepsilon := -\infty$ plays the role of the null element and $e := 0$ the role of the identity element. We refer the reader to [3] to learn the main properties of this structure. One essential feature is that $\oplus$ is idempotent which means that $x \oplus x = x, \forall x$. The combinatorial properties of dioids (*i.e.* associativity, commutativity of $\oplus$ and $\otimes$, and distributivity of $\otimes$ over $\oplus$), plus other familiar properties such that $\varepsilon \otimes x = \varepsilon, \forall x$, allow matrix manipulations in a conventional way. With these notations at hand, the third equation (1) now reads

$$x^3_n = x^1_n \oplus 1x^2_n \oplus 2x^3_{n-1}$$

In matrix notations, system (1) can be written in the form

$$x_n = A_0 x_n \oplus A_1 x_{n-1} \oplus B_0 u_n \oplus B_1 u_{n-1}; \quad y_n = C_0 x_n \oplus C_1 x_{n-1} \quad (2)$$

where it is left to the reader to construct the matrices.

This model is in the *event domain* since index n numbers events. Following the conventional approach, one can then introduce the formal backward shift operator in counting, namely γ such that $\gamma x^i_n = x^i_{n-1}$ and the γ-transform signals $X^i(\gamma) = \bigoplus_{n \in \mathbb{Z}} x^i_n \gamma^n$ (with the convention that $x^i_n = \varepsilon$ for n less than the origin of counting). $X^i(\gamma)$ is a Laurent series with coefficients in $\mathbb{Z}_{\max}$; let us denote this set by $\mathbb{Z}_{\max}\langle\langle\gamma\rangle\rangle$. Then from (2), one derives an implicit equation for the vector X [1], namely

$$X = (A_0 \oplus \gamma A_1) X \oplus (B_0 \oplus \gamma B_1) U$$

which is solved by $X = (A_0 \oplus \gamma A_1)^* (B_0 \oplus \gamma B_1) U$ with the notation $a^* := e \oplus a \oplus a^2 \oplus \dots$ (see [3]). Finally, one gets the input-output representation (transfer matrix)

$$Y = HU \text{ with } H(\gamma) = (C_0 \oplus \gamma C_1)(A_0 \oplus \gamma A_1)^* (B_0 \oplus \gamma B_1)$$

$\mathbb{Z}_{\max}\langle\langle\gamma\rangle\rangle$ is also a dioid with the addition and multiplication derived from those of $\mathbb{Z}_{\max}$ in the conventional way. But it is not the one that we need. Indeed, since firings are numbered in the order in which they occur, we must have $x_n \geq x_{n-1}$ which is equivalent to $x_n = x_n \oplus x_{n-1}$ implying that $X = X \oplus \gamma X$ which in turn implies that $X = \gamma^* X$. This leads us to consider the following equivalence relation

$$X(\gamma) \equiv X'(\gamma) \iff \gamma^* X(\gamma) = \gamma^* X'(\gamma)$$

The quotient of $\mathbb{Z}_{\max}\langle\langle\gamma\rangle\rangle$ by this equivalence relation is also a dioid denoted hereafter by $\mathcal{Z}_{\max}$ (this amounts to replacing x_n by $\sup_{m \leq n} x_m$). In addition to the conventional calculation rules of $\mathbb{Z}_{\max}\langle\langle\gamma\rangle\rangle$, we have the following simplification rule for monomials

$$t\gamma^n \oplus t\gamma^m = t\gamma^{\min(n,m)} \quad \text{(equality in } \mathcal{Z}_{\max}\text{)} \quad (3)$$

2.2 Counter approach

A dater function, say $n \mapsto d_n$ is generally noninvertible although it is monotone nondecreasing, since several events may occur at the same time or no event may occur at a specified time t. That is the equation $d_n = t$ may have one, several, or no solution in n. In such an instance, we may use, as a substitute for a solution, the concept of the "smallest oversolution", *i.e.*

$$c_t = \inf_{d_n \geq t} n \quad (4)$$

or that of the "largest undersolution", *i.e.*

$$\bar{c}_t = \sup_{d_n \leq t} n \quad (5)$$

In words, it means that we may adopt as the definition of the *counter* at time t "the smallest number over events that occur at or after t" or alternately "the largest number over the events that occur before or at t". It can be proved that

$$\bar{c}_t = c_{t+1} - 1 \quad (6)$$

Whichever definition we retain, it can be seen, by direct reasoning on the Petri net shown earlier, that counters satisfy similar equations as daters except that min replaces max, the delays on index t (now we operate in the *time domain*!) for a pair of transitions separated by a place is given by the holding time of the place, and the "coefficient" in $\mathbb{Z}_{\min} = (\mathbb{Z} \cup \{+\infty\}, \min, +)$ is given by the initial marking of that place. In $\mathbb{Z}_{\min}$ (now $\oplus$ has a different

[1] Upper case letters for vectors or scalars denote γ-transforms, *i.e.* X is a shorter notation for $X(\gamma)$

meaning), if for example x_t^i denotes the counter attached to transition x^i, we get

$$x_t^1 = 1x_t^2 \oplus u_{t-3}^1 \qquad x_t^2 = x_{t-1}^1 \oplus 1u_{t-1}^2$$
$$x_t^3 = x_t^1 \oplus x_{t-1}^2 \oplus 1x_{t-2}^3 \qquad y = 1x_t^2 \oplus x_{t-3}^3$$

Considering the formal backward shift operator δ in dating such that $\delta x_t = x_{t-1}$, we can code signals $\{x_t\}$ by their δ-transforms $\bigoplus_{t\in\mathbb{Z}} x_t\delta^t$ and convert the above equations in the conventional way to eventually get a transfer matrix $H(\delta)$ from the input vector $U(\delta)$ to the output vector $Y(\delta)$. The entries of these vectors lie in $\mathbb{Z}_{\min}\ll\delta\gg$, the set of Laurent series in δ with coefficients in $\mathbb{Z}_{\min}$. But again, we must "filter" all signals to consider only nondecreasing ones. The inequality $x_t \geq x_{t-1}$ now translates into $x_{t-1} = x_{t-1} \oplus x_t$ thanks to the new meaning of $\oplus$. This subsequently translates into $X = X \oplus \delta^{-1}X$ which implies that $X = \left(\delta^{-1}\right)^* X$. Finally, we are led to introduce the following equivalence relation

$$X(\delta) \equiv X'(\delta) \Longleftrightarrow \left(\delta^{-1}\right)^* X(\delta) = \left(\delta^{-1}\right)^* X'(\delta) \qquad (7)$$

Practically, it means that x_t is replaced by $\inf_{s\geq t} x_s$. The quotient of $\mathbb{Z}_{\min}\ll\delta\gg$ by this equivalence relation is a dioid denoted by $\mathcal{Z}_{\min}$. In addition to the conventional calculation rules for $\mathbb{Z}_{\min}\ll\delta\gg$, we have the additional simplification rule for monomials

$$n\delta^t \oplus n\delta^s = n\delta^{\max(t,s)} \quad \text{(equality in } \mathcal{Z}_{\min}) \qquad (8)$$

Notice that since a monomial $n\delta^t$ in $\mathcal{Z}_{\min}$ is coded as $t\gamma^n$ in $\mathcal{Z}_{\max}$, it is clear that the rule above simply translates into $t\gamma^n \oplus s\gamma^n = \max(t,s)\gamma^n$ which can be seen as a consequence of the fact that in $\mathcal{Z}_{\max}$, coefficients belong to $\mathbb{Z}_{\max}$. The dual comment could have been done for (3).

2.3 Two-dimensional domain

Because of the symmetrical roles played by the coefficients and the exponents in either $\mathcal{Z}_{\max}$ or $\mathcal{Z}_{\min}$, it is reasonable to introduce a two-dimensional domain representation. The two shift operators γ and δ are involved as formal variables in Laurent series with *boolean* coefficients: this set is denoted by $\mathcal{B}\ll\gamma,\delta\gg$. A monomial $t\gamma^n$ or $n\delta^t$ will simply be coded as $\gamma^n\delta^t$. In addition to the conventional addition and multiplication rules for series derived from those for boolean coefficients (in particular, addition is idempotent for series as it is for boolean numbers), we must draw the consequences of (3) and (8), namely that

$$\gamma^n\delta^t \oplus \gamma^n\delta^s = \gamma^n\delta^{\max(t,s)} \text{ and } \gamma^n\delta^t \oplus \gamma^m\delta^t = \gamma^{\min(n,m)}\delta^t \qquad (9)$$

Again this new structure is a dioid which can be interpreted as the quotient of $\mathcal{B}\ll\gamma,\delta\gg$ by the equivalence relation

$$X(\gamma,\delta) \equiv X'(\gamma,\delta) \Longleftrightarrow \gamma^* \left(\delta^{-1}\right)^* X(\gamma,\delta) = \gamma^* \left(\delta^{-1}\right)^* X'(\gamma,\delta) \qquad (10)$$

where the equality on the right-hand side is in $\mathcal{B}\ll\gamma,\delta\gg$. This quotient dioid is called MinMax$\ll\gamma,\delta\gg$ or simply $\mathcal{M}$ for short.

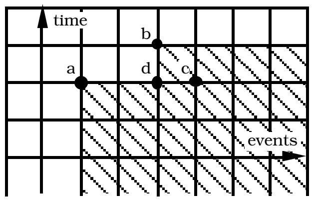
We refer the reader to [3] for the interpretation of an element $A = \bigoplus_{i\in A}\gamma^{n_i}\delta^{t_i}$ of $\mathcal{M}$ as an information set[2]. In short, a monomial $\gamma^n\delta^t$ is to be interpreted as the following piece of information:

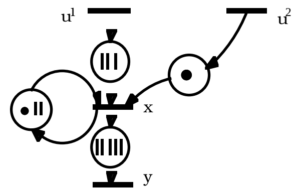
[2]For convenience, A will denote at the same time a subset of $\mathbb{Z}^2$ and a collection of indices i indexing the points of that subset.

"The event numbered n occurs at the earliest at time t."

This piece of information, say [b] on the picture below, is stronger than all other pieces of information lying in the south-east cone with vertex [b], as [d] and [c] (think of it!). This is the graphical translation of (9) or (10).

$\oplus$ in $\mathcal{M}$ amounts to $\cup$ for the corresponding information sets (hence a polynomial is a union of cones as in the picture above), $\otimes$ in $\mathcal{M}$ is the vector sum of sets, the null element ε in $\mathcal{M}$ is the polynomial with coefficients all null, which corresponds to the empty set, and the identity element e is the monomial $\gamma^0\delta^0$ which can also be represented as $\gamma^*\left(\delta^{-1}\right)^*$—see (10)—and which corresponds to the cone with vertex at the origin.





For the event graph shown earlier, it can be shown that the input-ouput 2-D representation reduces to

$$Y = \delta^5\left(\gamma\delta^2\right)^*\left(\delta^3U^1 \oplus \gamma U^2\right)$$

It corresponds to the above simpler event graph which nevertheless has exactly the same input-output behaviour (*i.e.* the same transfer matrix) as the previous one.

2.4 Passing from one representation to another

Suppose we are given an information set $A = \bigoplus_{i\in A}\gamma^{n_i}\delta^{t_i}$. We recover the corresponding dater function d_n^A by

$$d_n^A = \sup_{i\in A, n_i\leq n} t_i$$

Conversely, the set A can be obtained from the dater function by

$$A = \{(n_i,t_i)\in\mathbb{Z}^2 \mid t_i \leq d_{n_i}^A\} \qquad (11)$$

In Section 2.2, we proposed two possible definitions of the counter function, namely (4) and (5). We shall retain the former definition for reasons to be given later on in this section. Eq. (4) gives the way we pass from the dater to the counter function. Conversely, it can be proved that

$$d_n = \sup_{c_t\leq n} t$$

Now given an information set A, we have that

$$c_t^A = \inf_{i\in A, t_i\geq t} n_i$$

Conversely, the set A can be obtained from the counter function by

$$A = \{(n_i,t_i)\in\mathbb{Z}^2 \mid n_i \geq c_{t_i}^A\} \qquad (12)$$

This means that a piece of information (n,t) should be interpreted as

"Up to time t, the counter reaches at most the value n"

Finally, the three elements $\bigoplus_{i\in A}\gamma^{n_i}\delta^{t_i}$, $\bigoplus_{n\in\mathbb{Z}}\gamma^{n}\delta^{d_n^A}$ and $\bigoplus_{t\in\mathbb{Z}}\gamma^{c_t^A}\delta^{t}$ of $\mathcal{M}$ are the same as long as the above relationships hold. Had we taken (5) as the definition of the counter function, because of (6) we would have to consider $\bigoplus_{t\in\mathbb{Z}}\gamma^{\bar{c}_t^A+1}\delta^{t+1}$ instead of the previous expression with c_t^A, which would have made things rather complicated when multiplying such expressions by one another.

3 Numerical functions associated with an information set and its associated counter function

3.1 Support function of the information set

For any given subset $A \subset \mathbb{R}^n$, it is common to define the so-called support function (see [4]) as

$$\forall p \in \mathbb{R}^n, \quad \mathcal{S}_A(p) := \sup_{x\in A}\langle p, x\rangle \tag{13}$$

where $\langle .,. \rangle$ denotes the scalar product. On the other hand, $\mathcal{S}_A$ may be defined as the Fenchel conjugate of the indicator function of A defined as

$$\mathcal{I}_A(x) = \begin{cases} 0 \text{ if } x \in A \\ +\infty \text{ otherwise}\end{cases}$$

$$\mathcal{S}_A(p) = \max_{x\in\mathbb{R}^n}\left(\langle p, x\rangle - \mathcal{I}_A(x)\right)$$

$\mathcal{S}_A$ is convex as the upper hull of a family of affine functions. From a classical theorem (see [1, pp. Proposition 4.4]), if $\hat{X}(p)$ denotes the set of points x for which the supremum is achieved in (13) (if any), one has that

$$\partial\mathcal{S}_A(p) = \hat{X}(p) \tag{14}$$

where ∂f denotes the subdifferential of a convex function f. As a subdifferential, $\hat{X}(p)$ is a monotone multivalued operator, that is

$$\forall p_1, p_2, \forall x_1 \in \hat{X}(p_1), x_2 \in \hat{X}(p_2), \quad \langle p_1 - p_2, x_1 - x_2\rangle \geq 0 \tag{15}$$

In fact, $\mathcal{S}_A$ only characterizes the (closed) convex hull of A (denoted by $\overline{\text{co}}A$), the indicator function of which can be recovered from $\mathcal{S}_A$ by using the Fenchel conjugate once more. It also clear that $\mathcal{S}_A$ is positively homogeneous of degree one, that is

$$\forall \lambda \geq 0, \quad \mathcal{S}_A(\lambda p) = \lambda\mathcal{S}_A(p)$$

We can apply this to an information set A considered as a subset of either $\mathbb{R}^2$ or $\mathbb{Z}^2$, since this will make no difference as far as the support function is concerned. Let $p \in \mathbb{R}^2$ in (13) be equal to (γ, δ). Since A is unbounded in the south-east direction, it is obvious that

$$\gamma > 0 \text{ or } \delta < 0 \Rightarrow \mathcal{S}_A(\gamma, \delta) = +\infty$$

We have that

$$\mathcal{S}_A(\gamma, \delta) = \sup_{i\in A}\left(\gamma n_i + \delta t_i\right) \tag{16}$$

It means that the usual coding of A in $\mathcal{M}$, namely $\bigoplus_{i\in A}\gamma^{n_i}\delta^{t_i}$, is to be interpreted as a numerical function since, in the dioid notations where the product is the arithmetic sum, a monomial $\gamma^n\delta^t$ is the arithmetic expression $n\gamma + t\delta$; as for $\oplus$ of the formal power series (translating "union"), it is to be taken as max.

Since $\mathcal{S}_A$ is positively homogeneous, it suffices to know its values for all the values of the ratio $\sigma = -\delta/\gamma$ for $\sigma \in [0, +\infty]$ (remember that $\mathcal{S}_A$ is non trivial only for $\gamma \leq 0$ and $\delta \geq 0$). σ is the slope of the line $\gamma y + \delta x = 0$.

3.2 Numerical function associated with the counter function

Consider the counter function c^A associated with A. Using the description of A provided by (12) in (16) we get

$$\mathcal{S}_A(\gamma, \delta) = \sup_{n\geq c_t^A}(\gamma n + \delta t)$$

Since we are going to consider only nonpositive values of γ, n must be taken as small as possible, that is $n = c_t^A$. Hence we come up with the formula

$$\mathcal{S}_A(\gamma, \delta) = \sup_{t\in\mathbb{Z}}\left(\gamma c_t^A + \delta t\right)$$

Therefore, it turns out that $\mathcal{S}_A(-1, \delta) = \left(c^A\right)^*(\delta)$ where $\left(c^A\right)^*$ is exactly the Fenchel conjugate of function c^A. Indeed, it will appear more convenient to consider

$$\mathcal{C}_A(\delta) = -\mathcal{S}_A(-1, \delta) = -\left(c^A\right)^*(\delta) = -\sup_{t\in\mathbb{Z}}\left(\delta t - c_t^A\right)$$

which will be the numerical function associated with the counter function.

4 Basic operations on information and counters, and their numerical counterparts

The two basic operations of a dioid, addition and multiplication, correspond to the two basic operations of system theory consisting in cascading systems in parallel and in series. In this section, we examine how these two operations are related to our various representations, namely

1. information sets as particular subsets of $\mathbb{Z}^2$ which extend south-east;
2. indicator functions associated with those subsets;
3. $\mathcal{M}$-coding of those subsets:
4. support functions of those subsets;
5. counter functions;
6. their associated numerical functions.

Items 1, 2, 3, 5 lie in isomorphic dioids whereas items 4, 6 are in dioids which are isomorphic to each other but only homomorphic to the previous ones (that is there exist mappings from the former to the latter which are not bijective but which preserve addition and multiplication).

The following table summarizes the situation. The symbol $\Box$ denotes "inf-convolution" defined as [4] $(f\Box g)(x) := \inf_y f(x-y) + g(y)$. Here however, we need a "discrete" inf-convolution, that is the "inf" is taken over $\mathbb{Z}$. All "inf", "sup" and "+" operations concerning functions are pointwise operations. It is recalled that, $\forall A, \mathcal{S}_A(\gamma, \delta) = +\infty$ when either $\gamma > 0$ or $\delta < 0$ and $\mathcal{C}_A(\sigma) = -\infty$ when $\sigma < 0$.

		Sum	Zero	Product	Identity
1	A	$\cup$	$\varnothing$	$+$ (vector sum)	cone at the origin
2	$\mathcal{I}_A$	inf	$\mathcal{I}_A \equiv +\infty$	$\Box$	$\mathcal{I}_A(n,t) = 0$ iff $n \geq 0$ and $t \leq 0$
3	$\oplus\gamma^{n_i}\delta^{t_i}$	$\oplus$	ε (null series)	$\otimes$	$e = \gamma^0\delta^0$
4	$\mathcal{S}_A$	sup	$\mathcal{S}_A \equiv -\infty$	$+$	$\mathcal{S}_A(\gamma,\delta) = 0$ iff $\gamma \leq 0$ and $\delta \geq 0$
5	c^A	inf	$c^A \equiv +\infty$	$\Box$	0 for $t \leq 0$, $+\infty$ for $t > 0$
6	$\mathcal{C}_A$	inf	$\mathcal{C}_A \equiv +\infty$	$+$	$\mathcal{C}_A(\sigma) = 0$ for $\sigma \geq 0$

5 Line functions and their use in spectral analysis

5.1 Line functions as eigenfunctions

A "line function of slope σ" is a "best discrete approximate" of the linear function $y = \sigma x$ in the (t, n)-plane. Let us first introduce the following notation: for any real number a, $\lceil a \rceil$ will be the smallest integer number *larger* than or equal to a (*e.g.* $\lceil$-0.95$\rceil$=0). Then, a line function of slope σ—denoted by L_σ—is precisely the signal characterized by the counter function $c_t^{L_\sigma} = \lceil \sigma t \rceil, \forall t \in \mathbb{Z}$ (we assume that $\sigma \geq 0$). Pictorially, in the 2-D domain, *if time is still represented along the y-axis*—instead of the x-axis, then draw a line with slope $1/\sigma$ and keep all points of the $\mathbb{Z}^2$-grid to the *right* hand of this line. Notice that

$$\mathcal{C}_{L_\sigma}(\delta) = \begin{cases} 0 & \text{if } \delta = \sigma \\ -\infty & \text{otherwise} \end{cases}$$

Consider a SISO system with a transfer function H driven by an input U equal to some L_σ. Let $Y = HU$. From the above considerations, it should be clear that

$$\mathcal{C}_Y(\delta) = \begin{cases} \mathcal{C}_H(\sigma) & \text{if } \delta = \sigma \\ -\infty & \text{otherwise} \end{cases}$$

Using the usual $\otimes$ for + in dioid notations, this can also be summarized in the following form

$$\forall \delta \in \mathbb{R}, \quad \mathcal{C}_Y(\delta) = \mathcal{C}_H(\sigma) \otimes \mathcal{C}_U(\delta)$$

which shows that, in terms of the associated numerical functions, every line function behaves as an eigenfunction with respect to every transfer function. Going back from the numerical function to the original counter function involves some loss of information since only convex hulls can be characterized in this way. Therefore, we are going to establish a stronger result by manipulating the counter functions directly.

We have

$$\mathcal{C}_H(\sigma) = \inf_{\theta \in \mathbb{Z}} \left(c_\theta^H - \sigma\theta \right) = \nu - \sigma\tau \tag{17}$$

with $\nu = c_\tau^H$ for every τ belonging to the "arg inf", a set which depends of course on both H and σ. For the time being, we assume that this set is nonempty, that is the "inf" is reached, which happens whenever it is not equal to $-\infty$. We shall consider the case $\mathcal{C}_H(\sigma) = -\infty$ later on.

Then

$$\begin{aligned} c_t^Y &= \inf_{\theta \in \mathbb{Z}} \left(c_\theta^H + c_{t-\theta}^U \right) \\ &= \inf_{\theta \in \mathbb{Z}} \left(c_\theta^H + \lceil \sigma(t-\theta) \rceil \right) \\ &= \left\lceil \inf_{\theta \in \mathbb{Z}} \left(c_\theta^H - \sigma\theta \right) + \sigma t \right\rceil \\ &\quad \text{since } \lceil . \rceil \text{ is a monotone function and since } c_\theta^H \text{ is an integer} \\ &= \lceil \sigma(t-\tau) + \nu \rceil \quad \text{from (17)} \\ &= \lceil \sigma(t-\tau) \rceil + \nu \quad \text{since } \nu \text{ is integer} \end{aligned}$$

Finally

$$c_t^Y = \nu \otimes c_{t-\tau}^U \tag{18}$$

It is clear that every line input behaves as an eigenfunction for every transfer function and that the "phase shift" τ and the "amplification gain" ν are two integers which can be obtained from (17).

In $\mathcal{M}$, the same fact is expressed by the "eigenvalue-eigenvector" equation

$$H(\gamma, \delta)U(\gamma, \delta) = \gamma^\nu \delta^\tau U(\gamma, \delta)$$

where τ and ν are (non uniquely) defined by the equation $\mathcal{S}_H(-1, \sigma) = (-1)^\nu \sigma^\tau$ (r.h.s. in the dioid notations).

This is similar to what happens with functions $e^{j\omega t}$ in conventional Linear System Theory. Notice that transients are avoided here by letting our line functions start at $t = -\infty$, but a similar result could have been obtained for asymptotics if we had started them at $t = 0$. However, two remarks are in order at this point.

First, we do not claim that the family of line functions $\{L_\sigma\}_{\sigma \geq 0}$ forms a "basis" for all signals that are of interest for discrete event systems, as is the case for $\{e^{j\omega t}\}_{\omega \geq 0}$ in the conventional case (Fourier analysis). Hence knowing the response of a given system to each line function does not allow one to compute its response to other inputs in general. As already seen, $\mathcal{S}_H$ only characterizes the convex hull of the information set of the impulse response, which is sufficient for these line function inputs, but which will not be sufficient for more general inputs. More on this topic in a forthcoming paper.

Secondly, timed event graphs exhibit a "low-pass" effect, which was already well understood (see [2]). Namely, such systems[3] have their own "limit rate" under which periodic inputs can be normally processed, but over which only this typical limit rate can be observed at the output (and tokens accumulate inside). This translates into the fact that the infimum is equal to $-\infty$ in (17). More specifically, it should be clear that $\mathcal{C}_H$ is a nonincreasing function of σ and if

$$\lim_{t \to +\infty} c_t^H / t = \sigma_0 \tag{19}$$

then $\forall \sigma > \sigma_0, \mathcal{C}_H(\sigma) = -\infty$. In this case, we may consider in (17) that $\tau = +\infty$ (and $\nu = \lim_{t \to +\infty} c_t^H$). That is, the corresponding line functions are indefinitely delayed by the system. This never happens if $\sigma_0 = +\infty$, which corresponds to timed event graphs without circuits. Otherwise, σ_0 is given by the smallest ratio, among all circuits of the graph, of the total number of tokens over the total number of "bars" (holding time units) along those circuits (see [2] for the dual statement in the dater point of view).

5.2 Black and Bode plots

The Black plot in conventional system theory is the set of points whose coordinates are $(\varphi(\omega), \log r(\omega)) := (\arg H(j\omega), \log |H(j\omega)|)$ when ω varies. The Bode plot is the pair of plots $\varphi(\omega)$ and $\log r(\omega)$ against $\log \omega$. By analogy, we may here define the Black plot as the set of points $(\nu(\sigma), \tau(\sigma))$ as σ varies from 0 to $+\infty$, and the Bode plot as the pair of plots of $\nu(\sigma)$ and $\tau(\sigma)$ against $\log \sigma$. Notice that both $\nu(\sigma)$ and $\tau(\sigma)$ are additive— conventional addition—for systems in series, as it is also the case for $\varphi(\omega)$ and $\log r(\omega)$ in conventional system theory. Notice also that for the Black plot, $\tau(\sigma)$ which appears as the phase shift in the counter point of view, but which would appear as the amplification gain in the dater point of view,

[3] we still consider only SISO systems to make things simple

is put along the y-axis, as it is usually done for t in our representation.

Actually $\tau(\sigma)$ and $\nu(\sigma)$ are not uniquely defined by (17). They are multivalued but nondecreasing functions. To prove this, we notice that this pair is also a member of the set $\arg\max_{(n,t)\in H}(\gamma n + \delta t)$ when $\sigma = -\delta/\gamma$. Hence, recalling (15), one has that

$$(\nu(\sigma_1) - \nu(\sigma_2))(\gamma_1 - \gamma_2) + (\tau(\sigma_1) - \tau(\sigma_2))(\delta_1 - \delta_2) \geq 0$$

whenever $\sigma_1 = -\delta_1/\gamma_1$ and $\sigma_2 = -\delta_2/\gamma_2$. Then, if we pick $\gamma_1 = \gamma_2 = -1$ and $\delta_1 = \sigma_1$, $\delta_2 = \sigma_2$, we see that $\tau(\sigma)$ is a monotone multivalued function, whereas with $\delta_1 = \delta_2 = 1$ and $\gamma_1 = -1/\sigma_1$, $\gamma_2 = -1/\sigma_2$, we see the same thing for $\nu(\sigma)$.

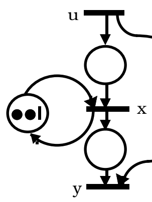
With these definitions, it is clear that the Black plot is simply the set of all points of $\mathbb{Z}^2$ lying on the border of the convex hull of the information set relative to the transfer function H. As for the Bode plot, it is a pair of staircase multifunctions. The points where these are truly multivalued correspond to the slopes that appear in the above mentioned convex hull.

5.3 Example

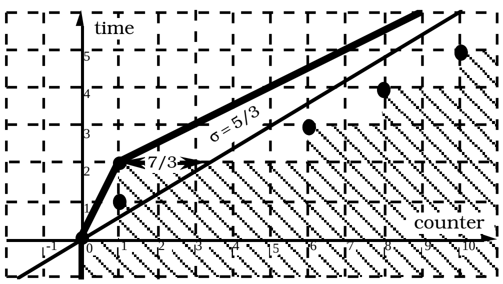
To close this section, let us examine the following example

$$H(\gamma,\delta) = \gamma\delta^2 \oplus \left(\gamma^2\delta\right)^*$$

corresponding to the Petri net on the right. The next picture shows the graph of c^H, the shaded area corresponding to the information set, and the solid line delineating its convex hull.



Observe that σ_0 as defined by (19) is equal to 2 (look at the circuit of the graph).



All points of the $\mathbb{Z}^2$ grid lying on the broken solid line belong to the Black plot. On this picture, the line function L_σ with $\sigma = 5/3$ is represented by the straight line. It can be checked that

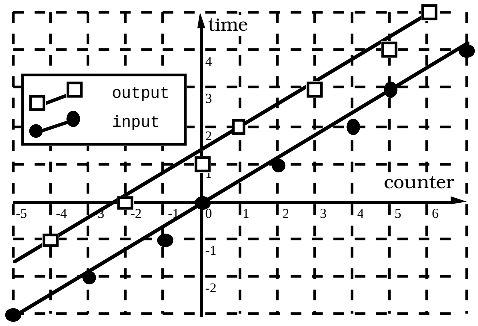
$$H(\gamma,\delta)\bigoplus_{t\in\mathbb{Z}}\gamma^{[5t/3]}\delta^t = \gamma^1\delta^2\bigoplus_{t\in\mathbb{Z}}\gamma^{[5t/3]}\delta^t$$

For $\sigma = 5/3$, the shifts $\nu = 1$ and $\tau = 2$ are explained by the picture. On this same picture, it can be seen that $\mathcal{C}_H(5/3) = -7/3$.

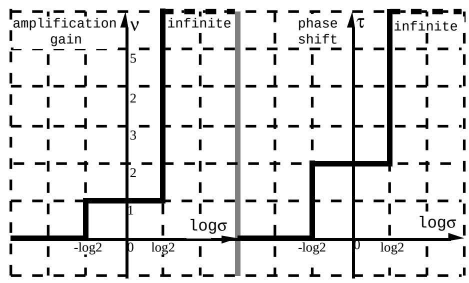
On the next drawing, the input $U = L_{5/3}$ and its corresponding shifted output Y are depicted. The "horizontal" difference $c^Y_t - c^U_t$ fluctuates between -2 during 2 time units over 3, and -3 for the remaining 1 over 3. The average is thus $-(2\times 2/3) - (3\times 1/3) = -7/3 = \mathcal{C}_H(5/3)$. This observation can be explained as follows. From (18), we have that

$$c^Y_t - c^U_t = [\sigma(t-\tau)] + \nu - [\sigma t]$$

It should be clear that the average value of the right-hand side is $\nu - \sigma\tau$ which is precisely equal to $\mathcal{C}_H(\sigma)$ according to (17).



Finally, the pair of Bode plots are shown below.



6 Conclusion

In conclusion, we offer the following table which should be self-explanatory.

min-linear systems	conventional linear systems
min, $+$	$\int$, $\times$
$c^Y_t = \inf_{\tau\in\mathbb{Z}}\left(c^H_{t-\tau} + c^U_\tau\right)$	$y(t) = \int_{\mathbb{R}} h(t-\tau)u(\tau)\,d\tau$
$c^{L_\sigma}_t = [\sigma t]$	$x_\omega(t) = e^{j\omega t}$
$\mathcal{C}_H(\sigma) = \inf_{t\in\mathbb{Z}}\left(c^H_t - \sigma t\right)$	$H(j\omega) = \int_{\mathbb{R}} h(t)e^{-j\omega t}\,dt$
$\mathcal{C}_Y(\sigma) = \mathcal{C}_H(\sigma) + \mathcal{C}_U(\sigma)$	$Y(s) = H(s)U(s)$
$H(\gamma,\delta)L_\sigma(\gamma,\delta) = \gamma^{\nu(\sigma)}\delta^{\tau(\sigma)}L_\sigma(\gamma,\delta)$	$H(s)X_\omega(s) = e^{\log r(\omega)}e^{j\varphi(\omega)}X_\omega(s)$

References

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[4] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.