Fundamental Traffic Diagrams : A Maxplus Point of View

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14/6/2010

Outline

Introduction

Main Points

- Following Daganzo we discuss the variational formulation of the Lighthill-Witham-Richards equation describing the traffic on a road.
- We extend it to the case of two roads with a junction with the right priority.
- The equation obtained is no more an HJB equation. To study its eigenvalue we consider its space discretization for which we are able to derive analytically its eigenvalue as function of the car density. This function gives a good approximation of what we call the global fundamental traffic diagram.

Lighthill-Whitham-Richards Model

The LWR Model expresses the mass conservation of cars :

$$\begin{cases} \partial_t \rho + \partial_x \varphi = 0 \ , \\ \varphi = f(\rho), \end{cases}$$

- $\varphi(x,t)$ denotes the flow at time t and position x on the road.
- $\rho(x,t)$ denotes the density.
- $f(\rho)$ is a given function called the *fundamental traffic diagram*.

For traffic, this diagram plays the role of the gas law for the fluid dynamics.
The diagram has been estimated using experimental data, and its behavior is quite different from standard gas at high density.

Variational Formulation of the LWR Model

Adapting Daganzo in [4].

- q_x^t denotes the cumulated number of vehicles having reached the point x at time t.
- $\varphi = \partial_t q$ is the flow.
- $\rho = -\partial_x q + a$ is the car density where a is the initial density.

If f is concave, it exists f^* convex : $f(\rho) = \inf_u\{-u\rho + f^*(u)\}$.

$$\begin{cases} \varphi = f(\rho), \\ \partial_t \rho + \partial_x \varphi = 0, \end{cases} \iff \begin{cases} \partial_t q = \inf_u\{u(\partial_x q - a) + f^*(u)\}, \\ \partial_{tx} q = \partial_{xt} q. \end{cases}$$

Variational Formulation of the Traffic on a Line

The LWR model is equivalent to a control model where the control is the drift term and the cost function is the dual of the opposite of the fundamental traffic diagram.

Global Fundamental Diagram

The HJB equation admits eigen-elements λ and r_x satisfying :

$$\lambda = \inf_u \{u\partial_x r + f^*(u))\}, \forall x.$$

This eigenvalue is equal to the average growth rate :

$$\chi = \lim_{T \to +\infty} \frac{q_x^T - q_x^0}{T}, \forall x.$$

Moreover if the road (R) is closed there is car conservation in the system, that is, there exists a constant d (the global density):

$$d = 1/|R| \int_R (-\partial_x q^t + a)dx = 1/|R| \int_R a dx, \ \forall t.$$

Global Fundamental Traffic Diagram

When it exists $\bar{f}$ such that $\chi = \bar{f}(d)$ we call $\bar{f}$ the global fundamental traffic diagram.

Circular Road

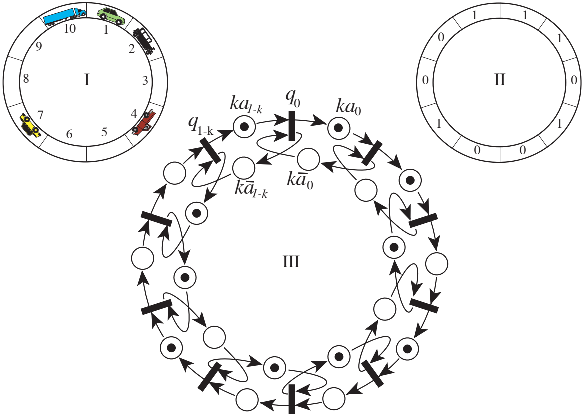


Figure: Petri net representation of the traffic on a circular roads.

Traffic Modeling on a Circular Road

- The vehicle number entered in the section x before time t is denoted q_x^t.
- The initial vehicle quantity [resp. available place] in the section $[x, x+k]$ is $a_x k$. [resp. $\bar{a}_x k$ with $\bar{a}_x = (1 - a_x)$].
- The index x being modulo 1, the dynamics is given by :

$$q_x^{t+h} = \min\{a_{x-k}k + q_{x-k}^t, \bar{a}_x k + q_{x+k}^t\} ,$$

The speed of cars being $v = 1$ and choosing $k = h$, with $h \to 0$ we obtain :

Circular Road Traffic Equation :

$$\partial_t q = \min\{-\partial_x q + a, \partial_x q + \bar{a}\}.$$

Fundamental Traffic Diagram

Since in the circular road case :

$$f^*(u) = \begin{cases} 0 \text{ when } u = -1, \\ 1 \text{ when } u = 1, \end{cases}$$

we have :

Circular Road Fundamental Traffic Diagram:

$$f(\rho) = \min\{\rho, 1 - \rho\}.$$

Global Fundamental Traffic Diagram

Theorem

The global fundamental traffic diagram of the circular road is :

$$\chi(d) = \min(d, 1-d),$$

where $d = \int_0^1 a_x dx$ *is the average car density on the road.*

Proof: There three way to make a circuit on the road using the two possible control $\{-1, 1\}$ of the traffic HJB equation :

- Always backward $u = 1$ with average cycle cost : $\int_0^1 a_x dx = d$.
- Always forward $u = -1$ with average cycle cost : $\int_0^1 (1 - a_x) dx = 1 - d$.
- Backward $u = 1$ during τ, forward during τ : $1/(2\tau)[\int_y^{y+\tau} a_x dx + \int_y^{y+\tau} (1 - a_x) dx] = 1/2$.

Then the result follows from the fact $1/2 \geq \min\{d, 1-d\}$.

A height shape road with right priority

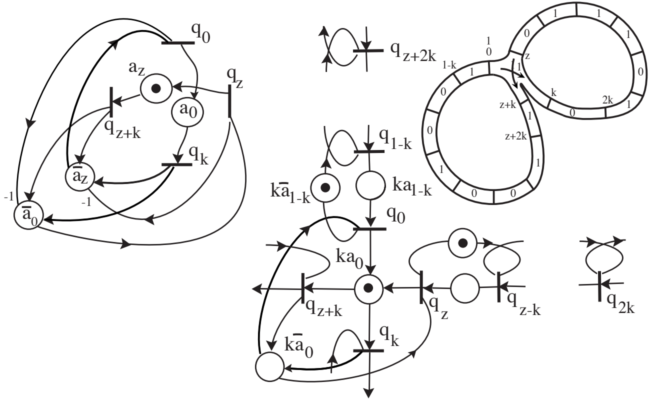
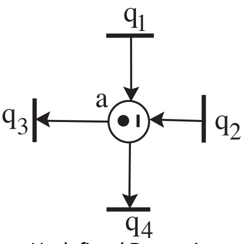


Figure: A road having the height shape cut in sections (top-right), its Petri net simplified modeling (middle) and the precise modeling of the junction (top left).

Conflict resolution using negative weights 1



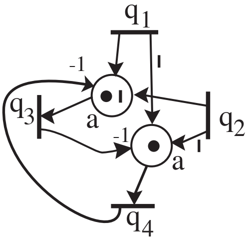


Figure: The Petri net with conflict, given in the left figure, is made clear by giving top priority to q_3 against q_4 in the right figure.

Conflict resolution using negative weights 2

The constraint expressed by the Peti net with conflict is :

$$q_4^k q_3^k = a q_1^{k-1} q_2^{k-1} \; .$$

clearly q_3 and q_4 are not defined uniquely. To define them uniquely we can clarify the dynamics by : Choose a priority rule (top priority to q_3 against q_4)

$$q_3^k = a q_1^{k-1} q_2^{k-1} / q_4^{k-1}, \quad q_4^k = a q_1^{k-1} q_2^{k-1} / q_3^k \; .$$

Height Shape Road Dynamics

We precise the dynamics of the traffic system by giving the right priority to enter in the junction.

$$\begin{cases} q_x^{t+h} = \min\{ka_{x-k} + q_{x-k}^t, k\bar{a}_x + q_{x+k}^t\}, \ x \neq z, 0 \ , \\ q_z^{t+h} = \min\{k\tilde{a}_z + q_k^t + q_{z+k}^t - q_0^{t+h}, ka_{z-k} + q_{z-k}^t\} \ , \\ q_0^{t+h} = \min\{k\tilde{a}_0 + q_k^t + q_{z+k}^t - q_z^t, ka_{1-k} + q_{1-k}^t\} \ , \end{cases}$$

This dynamics is homogeneous of degree 1 but not monotone. When the car speed $v = 1$, taking $k = h$, with $h \to 0$ we obtain the height shape road dynamics (HSRD) :

$$\begin{cases} \partial_t q = \min\{a - \partial_x^- q, \bar{a} + \partial_x^+ q\}, \ \forall x \neq 0, z \ , \\ (\partial_t q)_z = \min\{a_z - (\partial_x^- q)_z, \tilde{a}_z + (\partial_x^+ q)_0 + (\partial_x^+ q)_z - (\partial_t q)_0\}, \\ (\partial_t q)_0 = \min\{a_0 - (\partial_x^- q)_0, \tilde{a}_0 + (\partial_x^+ q)_0 + (\partial_x^+ q)_z\}, \end{cases}$$

This equation is not an HJB equation since the monotony is lost.

The Eigenvalue Problem

We want solve the eigenvalue problem find λ real and $r(x)$ such that :

$$\begin{cases} \lambda = \min\{a - \partial_x^- r, \bar{a} + \partial_x^+ r\}, \ \forall x \neq 0, z \ , \\ \lambda = \min\{a_z - (\partial_x^- r)_z, \tilde{a}_z + (\partial_x^+ r)_0 + (\partial_x^+ r)_z - \lambda\}, \\ \lambda = \min\{a_0 - (\partial_x^- r)_0, \tilde{a}_0 + (\partial_x^+ r)_0 + (\partial_x^+ r)_z\}, \end{cases}$$

In this case the growth rate is not equal to the eigenvalue but the eigenvalue gives good approximation of the growth rate for the discrete model.

Reducing the positive eigenvalue problem to an ergodic HJB equation

In the traffic problem we have $\tilde{a}_z = \tilde{a}_0 = 1 - a_0 - a_z$ since these two quantities mean the density of free place in the junction. Under the assumption that r is continuous at 0, if $\lambda > 0$ we have :

$$\lambda \leq \tilde{a}_z + (\partial_x r)_0 + (\partial_x r)_z - \lambda < \tilde{a}_z + (\partial_x r)_0 + (\partial_x r)_z$$

which implies that the research of the positive eigenvalue, when r is regular in 0, is reduced to solve :

$$\begin{cases} \lambda = \min\{a - \partial_x r, \bar{a} + \partial_x r\},\ \forall x \neq 0, z\ , \\ \lambda = \min\{a_z - (\partial_x r)_z, 1/3(\bar{a}_z + (\partial_x r)_z)\}, \\ \lambda = a_0 - (\partial_x r)_0. \end{cases}$$

Since this equation is an HJB ergodic equation, looking at the circuit in this system we deduce $\lambda = \min\{d, 1/4\}$.

Solving the Eigenvalue of the Discrete Model I

Using minplus notations we have to solve :

$$\begin{cases} \mu r_x = \alpha_{x-k} r_{x-k} \oplus \bar{\alpha}_x r_{x+k}, \ x \neq z, 0 \ , \\ \mu r_z = \bar{\alpha} r_k r_{z+k} / r_0 \mu \oplus \alpha_{z-k} r_{z-k} \ , \\ \mu r_0 = \bar{\alpha} r_k r_{z+k} / r_z \oplus \alpha_{1-k} r_{1-k} \ , \end{cases}$$

with $k = h$, $\mu = \lambda^k$, $\bar{\alpha} = (\tilde{a}_z)^k = (\tilde{a}_0)^k$ and $\alpha_x = (a_x)^k$. From

$$\mu r_z \leq \bar{\alpha} r_k r_{z+k} / r_0 \mu, \ \ \mu r_k \leq \alpha_0 r_0, \ \ \mu r_{z+k} \leq \alpha_z r_z,$$

we deduce $\lambda \leq 1/4$.

Solving the Eigenvalue of the Discrete Model II

From the first equation using $\lambda \leq 1/4$ we obtain :

$$r_x = \begin{cases} b_x r_k/\lambda^{x-k} \oplus \bar{b}_x r_z/\lambda^{z-x} \text{ if } 0 < x < z, \\ c_x r_{z+k}/\lambda^{x-z-k} \oplus \bar{c}_x r_0/\lambda^{1-x} \text{ if } z < x < 1, \end{cases}$$

with $b_x = \prod_k^{x-k} \alpha_s$, $\bar{b}_x = \prod_x^{z-k} \bar{\alpha}_s, c_x = \prod_{z+k}^{x-k} \alpha_s$, $\bar{c}_x = \prod_x^{1-k} \bar{\alpha}_s$.

$$\begin{cases} r_z = \bar{\alpha} r_k r_{z+k}/r_0 \lambda^{2k} \oplus b_z r_k/\lambda^z \ , \\ r_{z+k} = \alpha_z r_z/\lambda^k \oplus \bar{c}_{z+k} r_0/\lambda^{1-z} \ , \\ r_0 = \bar{\alpha} r_k r_{z+k}/r_z \lambda^k \oplus c_1 r_{z+k}/\lambda^{1-z} \ , \\ r_k = \alpha_0 r_0/\lambda^k \oplus \bar{b}_k r_z/\lambda^z \ , \end{cases}$$

Solving the Eigenvalue of the Discrete Model III

Searching for $(\lambda > 0)$ using

$$r_0 r_z \leq \bar{\alpha} r_k r_{z+k}/\lambda^{2k} < \bar{\alpha} r_k r_{z+k}/\lambda^k$$

the system can be reduced to :

$$\begin{cases} r_z = (\bar{\alpha}\lambda^{1-z-2k}/c_1 \oplus b_z/\lambda^z) r_k \ , \\ r_{z+k} = (\alpha_z/\lambda^k) r_z \oplus (\bar{c}_{z+k}/\lambda^{1-z}) r_0 \ , \\ r_0 = (c_1/\lambda^{1-z}) r_{z+k} \ , \\ r_k = (\alpha_0/\lambda^k) r_0 \oplus (\bar{b}_k/\lambda^z) r_z \ , \end{cases}$$

This system being linear in r we have to find the λ such that the circuits of minimal weight are 0.

Precedence graph

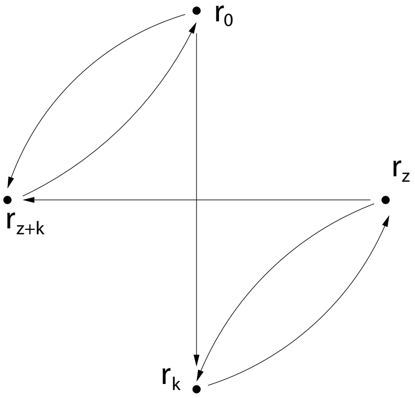


Figure: Precedence Graph of the Eigenvalue System.

Eigenvalue of the eight shape road

There are three circuit :

- (r_k, r_z, r_k) with weight :

$$(\bar{\alpha}\lambda^{1-z-2k}/c_1 \oplus b_z/\lambda^z)(\bar{b}_k/\lambda^z).$$

- (r_{z+k}, r_0, r_{z+k}) with weight :

$$(\bar{c}_{z+k}/\lambda^{1-z})(c_1/\lambda^{1-z}).$$

- $(r_k, r_z, r_{z+k}, r_0, r_k)$ with weight :

$$(\bar{\alpha}\lambda^{1-z-2k}/c_1 \oplus b_z/\lambda^z)(\alpha_z/\lambda^k)(c_1/\lambda^{1-z})(\alpha_0/\lambda^k).$$

Then passing to the limit $k \to 0$, we obtain :

Positive eigenvalue of the eight shape road :

$$\lambda = \min\{d, 1/4, (z-d)/(2z-1)\}.$$

Global fundamental traffic diagram

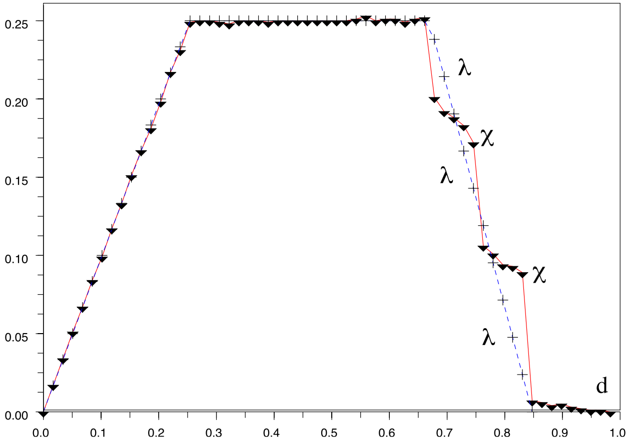


Figure: Comparison of the global fundamental traffic diagram and the eigenvalue.

Extension to Regular Towns

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LWR Equation

Circular Road

Height shape Road

Extension

Bibliography

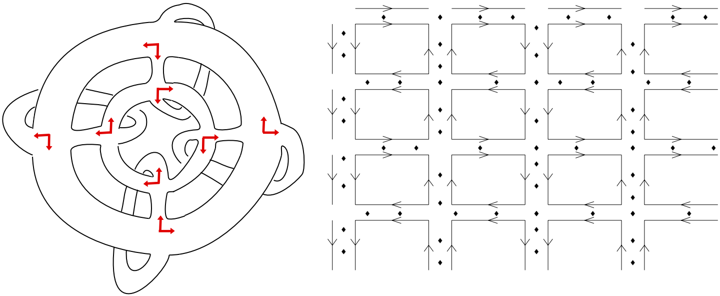

Figure: Roads on a torus of 4×2 streets with its authorized turn at junctions (left) and the asymptotic car repartition in the streets on a torus of 4×4 streets obtained by simulation.

Light Controlled Fundamental Diagram

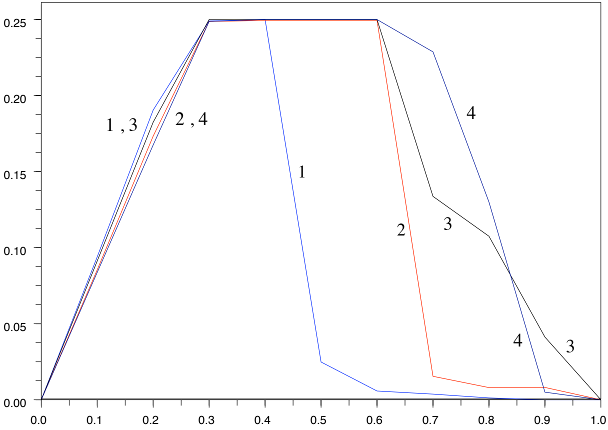

Figure: Light policies comparison for a regular town on a torus. (1) right priority, (2) open loop, (3) local feedback, (4) global feedback obtained by LQ methods.

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