CHAPTER 7

Application of stochastic control methods to the management of energy production in New Caledonia

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1. Introduction

The aim of this paper is to illustrate the use of stochastic control methods with a concrete and simple example taken from the management of the electricity production system of New Caledonia. The system consists of eight power plants and a dam. For this system we address the problem of minimizing the cost of meeting a given electricity demand. The variables to be optimized are the starting time of power plants and the quantities of turbined water. The system is complicated by many stochastic phenomena, including uncertainty in electricity consumption and input of water to the dam, and breakdown of power plants. We model these stochastic phenomena and solve the resulting optimization problem by the dynamic programming method.

From a practical point of view all the above phenomena are not equally important which leads us to decompose the problem into two parts:

(1) a long-run problem giving the management policy for the dam, the purpose of which being to give a final target cost for a short-run problem; and

(2) a short-run problem which models all the phenomena just described.

Applied Stochastic Control in Econometrics and Management Science, edited by A. Bensoussan et al.

Thus, the complete solution of the problem is decoupled into four parts:

(1) identification of the stochastic processes which appear in the long-run model;

(2) optimization of the long-run model;

(3) identification of the stochastic processes which appear in the short-run model; and

(4) optimization of the short-run model.

Indeed, the probabilistic characteristics of a stochastic process can change according to the time-scale taken, and this leads us to model the same stochastic processes in a different manner in the next two sections which present the long- and short-run models of interest.

2. Long-run model (horizon one year)

We now describe the long-run problem; its purpose is to give the long-run management of the dam. We use the following notation (dimensions in parenthesis):

t = time (T)
T = economic horizon (T)
X_t = real cumulated water supply up to time t (L^3)
A_t = transformed cumulated water supply up to time t (L^3)
β_t = flow of average supply, storms excluded (L^3/T)
$\nu(t, \mathrm{d}u) \times \mathrm{d}t$ = probability of a storm of intensity between u and $u + \mathrm{d}u$, between the instants t and $t + \mathrm{d}t$
S_t = stock of water in the dam at time t (L^3)
S_M = maximal storage capacity of the dam (L^3)
S_m = minimal water storage required for the dam (L^3)
u_t = flow of water turbined at time t (L^3/T)
u_M = maximal flow turbined (L^3/T)
u_m = minimal flow turbined (L^3/T)
$\mathrm{E}(S, u)$ = hydroelectric power produced when the stock is S and the flow turbined is u (ML^2T^{-3})
D_t = hydroelectric demand at time t (ML^2T^{-3})
$B(D)$ = mean variation of the demand per unit of time, for a given demand D (ML^3T^{-4})
$\sigma^2(D)$ = variance of the demand per unit of time, for a given demand D ($M^2L^4T^{-7}$)
W_t = Brownian motion perturbing the demand ($T^{1/2}$)

P_t = thermal power demanded at time t (ML^2T^{-3})

$P_{\min}$ = 0

P_{moy} = maximal power of thermal power plants of type 1 (ML^2T^{-3})

$P_{\max}$ = maximal power of all the thermal power plants, the gaz turbine excluded (ML^2T^{-3})

$C(P)$ = cost of production of the thermal power P per unit of time ($/T$)

ρ_1 = marginal cost of production of thermal power plants of type 1 ($\$M^{-1}L^{-2}T^2$)

ρ_2 = marginal cost of production of thermal power plants of type 2 ($\$M^{-1}L^{-2}T^2$)

ρ_3 = marginal cost of production of the gaz turbine ($\$M^{-1}L^{-2}T^2$)

$V(t, S, D)$ = Belman optimal return function for the long run problem.

The long-run model consists in the management of the following means of production (in such a way to meet the demand D_t for all t, with the minimum possible cost): four thermal power plants of type 1 (36 MW); four thermal power plants of type 2 (16 MW); one gas turbine (19 MW); and one dam equipped with four turbines of 16 MW.

The data are the statistics on: (a) the demand for electricity, and (b) the flow of water input to the dam: figs. 7.5–7.6

The outputs of the long-run model are:

(1) the flow of water turbined expressed in feedback form as a function of the stock of water in the dam and the demand for electricity;

(2) the power produced by both types of thermal power plants as a feedback function of the stock of water and the demand for electricity; and

(3) the optimal cost as a function of the stock of water and the demand for electricity.

Using the preceding notation we can write the state equations of the system. The demand for electricity is given by the stochastic differential equation

$$dD_t = b(D_t)\,dt + \sigma(D_t)\,dw_t. \tag{1}$$

The two functions b and σ^2 give the probability law of D.

The stock of water evolves according to:

$$dS_t = \begin{cases} -(dA_t - u_t\, dt)^-, & S_t = S_M, \\ dA_t - u_t\, dt, & S_m < S_t < S_m, \\ (dA_t - u_t\, dt)^+, & S_t = S_m, \end{cases} \tag{2}$$

$$u_m \leqslant u_t \leqslant u_M, \tag{3}$$

$$A_t = \int_0^t \beta(s)\, ds + \sum_{s<t} A_s - A_{s-}. \tag{4}$$

$A_s - A_{s-}$ represents a jump in the flow caused by a storm.

A_t is constructed from the real cumulated water supply X_t in the following manner:

$$A_t = \int_0^t \left(\frac{dX}{dt} \wedge u_M\right) ds + \sum_{T_n<t} (X_{T_n} - X_{T_n'} - u_M(T_n - T_{n'})), \tag{5}$$

where $x \wedge y = \min(x, y)$ and the sequence of stopping times T_n and T_n' are defined by

$$T_n = \inf\left\{t;\ t > T_n';\ \frac{dX_t}{dt} < u_M\right\}, \tag{6}$$

$$T_n' = \inf\left\{t;\ t > T_n;\ \frac{dX_t}{dt} > u_M\right\} \tag{7}$$

(see fig. 7.1).

The probability law of A_t will be characterized by: (a) the function $\beta(t)$: mean water supply (storms excluded), and (b) the probability law of the storms: $\nu(t, du)\, dt$ giving the probability of a storm of intensity between $(u, u + du)$, during the interval $(t, t + dt)$.

The cost function is the mathematical expectation of the cost of meeting the demand:

$$\min_{u_t} \mathrm{E} \int_0^T C(D_t - E(S_t, u_t))\, dt, \tag{8}$$

where C is a piecewise-linear function, increasing, convex. Its derivative is given by fig. 7.1, where the following points should be noted.

(1) The interval $[P_{\min}, P_{\text{moy}}]$ corresponds to the case where the thermal power is supplied by the thermal power plants of type 1.

(2) The interval $[P_{\text{moy}}, P_{\max}]$ corresponds to the case where thermal power is supplied by the thermal power plants of type 1 working at their maximum power and the remaining part being supplied by the thermal power plants of type 2.

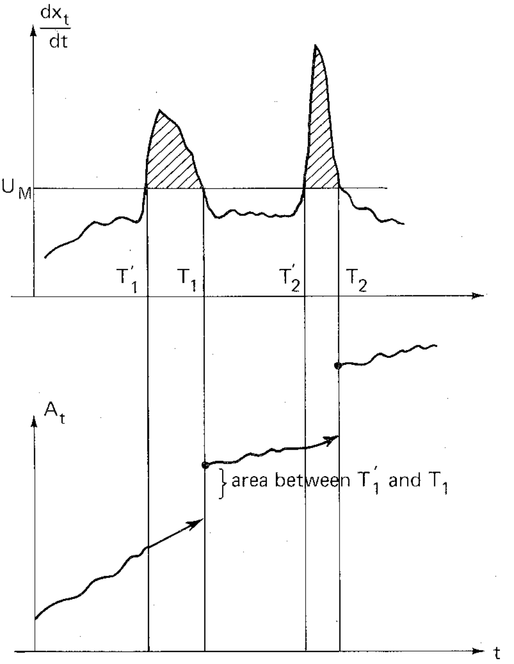

Figure 7.1

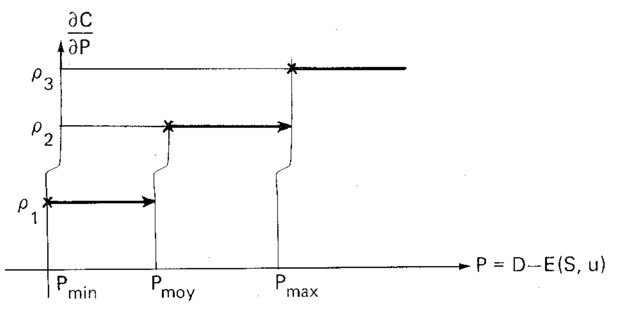

Figure 7.2

(3) The interval $[P_{max}, \infty]$ corresponds to the case where all the thermal power plants work at their maximum level, the remaining part being supplied by the gas turbine if possible otherwise the price of failing is supposed to be the price of production by the gas turbine.

3. Short-run model

We now describe a more detailed model of the system used to determine the starting time of the power plant knowing the long-run management of the dam. Let us first agree on the additional following notations:

M_t = number of thermal power plants of type 1 in activity at time t

$\lambda^- \,dt$ = probability of a breakdown between t and $t + dt$ for a thermal power plant of type 1

$\lambda^+ \,dt$ = probability that a thermal power plant of type 1 can be repaired between t and $t + dt$

N_t = number of thermal power plants of type 2 in operation at time t

N_t^1 = number of thermal power plants of type 2 wanted (power plant asked if no breakdown appears at this time) to be in operation at time t

N_t^2 = total number of breakdowns of thermal power plants of type 2 before the time t

$\mu \,dt$ = probability of a breakdown of a thermal power plant of type 2 between t and $t + dt$

P_t^1 (resp. P_t^2) = power supplied by the thermal power plants of type 1 (resp. of type 2) at time t (ML^2T^{-3})

P_m^1 (resp. P_m^2) = minimal power of a working thermal power plant of type 1 (resp. of type 2) (ML^2T^{-3})

P_M^1 (resp. P_M^2) = maximal power of a working thermal power plant of type 1 (resp. of type 2) (ML^2T^{-3})

$P_{min}(M, N)$ = minimal thermal power supplied by M thermal power plants of type 1 and N thermal power plants of type 2 (ML^2T^{-3})

$P_{moy}(M, N)$ = thermal power supplied by M thermal power plants of type 1 working at their maximal level and N of type 2 working at their minimal level (ML^2T^{-3})

$P_{max}(M, N)$ = maximal power supplied by M thermal power plants of type 1 and N of type 2 (ML^2T^{-3})

k = cost of starting up a thermal power plant of type 2 (\$)
$\tilde{\beta}(t)$ = mean flow of "short-run" water supply (L^3T^{-1})
$\tilde{\sigma}^2(t)$ = variance of the "short-run" water supply per limit of time (L^6T^{-1}).

The short-run model consists in managing the means of production described above, taking into account the starting decisions for the available thermal power plants. This problem is also studied in Leguay (1975).

The states of the system are: the stock of water in the dam S_t; the demand for electricity D_t; the number of thermal power plants of type 1 in operation M_t; and the number of thermal plants of type 2 in operation N_t.

The control variables are: the flow of water turbined; the thermal power produced by the thermal power plants of type 1 and 2; and the decisions to start and stop the thermal power plants of type 2.

Remark 7.1. For the thermal power plants of type 1, the following policy is used: every thermal power plant of type 1 available is immediately started. All the controls will be optimized in feedback form on the four state variables. The state equations are:

$$dA_t = \tilde{\beta}(t)\,dt + \tilde{\sigma}(t)\,dw_t, \tag{9}$$

$$dS_t = \begin{cases} -(dA_t - u_t\,dt)^-, & S_t = S_M, \\ dA_t - u_t\,dt, & S_m < S_t < S_M, \\ (dA_t - u_t\,dt)^+, & S_t = S_m, \end{cases} \tag{10}$$

$$u_m \leqslant u_t \leqslant u_M, \tag{11}$$

$$dD_t = b(D_t)\,dt + \sigma(D_t)\,dw_t. \tag{12}$$

M_t is a point process with values in the set $\{3, 4\}$. If we denote by $\lambda_{i,j}\,dt$ the probability that M_t goes from i to j between t and $t + dt$, we have

$$\begin{aligned} \lambda_{34} &= \lambda^+, \\ \lambda_{43} &= \lambda^-, \\ \lambda_{ij} &= 0, \qquad (i,j) \neq (3,4), (4,3), \end{aligned} \tag{13}$$

$$\begin{aligned} \Delta N_t &= -\Delta N_t^2 \text{ between the "starting" decisions,} \\ N_t &= N_t^1(N_{t^-} - \Delta N_t^2) \text{ at the instant of decision.} \end{aligned} \tag{14}$$

N_t^1 represents the state desired at time t, knowing that it was in state N_{t^-} at "time t^-" and knowing ΔN_t^2. ΔN_t^2 is 1 if there is a breakdown at time t, 0 otherwise; the probability of a breakdown being $N_{t^-}\mu\,dt$.

Remark 7.2. The modelling of water supply has been changed: one supposes here that it is a diffusion process. Indeed, the storms create sudden increases of the flow, these increases can be modelled by discontinuities in the long-run (time step of one week), but in the short run (time step of a few hours) a continuous model seems better. One supposes that the coefficients $\tilde{\beta}$ and $\tilde{\sigma}$ are given each week by the local weather forecast.

Moreover, one has the following constraints on the working of thermal power plants:

$$MP_{\mathrm{m}}^1 \leqslant P_t^1 \leqslant MP_{\mathrm{M}}^1, \tag{15}$$

$$NP_{\mathrm{m}}^2 \leqslant P_t^2 \leqslant NP_{\mathrm{M}}^2. \tag{16}$$

The problem consists in minimizing the cost function over a period of one week:

$$\min \mathrm{E}\int_0^T \left\{\rho_1 P_t^1 + \rho_2 P_t^2 + \rho_3\left(D_t - P_t^1 - P_t^2 - \mathrm{E}(S_t, u_t)\right)^+\right\} \mathrm{d}t$$
$$+ k \sum_{t<T} \left(N_t^1 - N_{t^-} + \Delta N_t^2\right)^+ + V(T, D_T, S_T). \tag{17}$$

The control variables are P^1, P^2, u, and N^1. The term

$$k \sum_{t<T} \left(N_t^1 - N_{t^-} + \Delta N_t^2\right)^+$$

represents the cost of starting up the thermal power plants. $V(T, D_T, S_T)$ is the terminal cost, which is the result of the long-run optimization problem.

4. Identification of the stochastic processes arising in both models

In this section we use the results of F. Delebecque and J. P. Quadrat (1978a)

4.1. *Identification of the demand for electricity in the short-run model*

We have available the value taken by this process each hour during one year (see fig. 7.1). The process is assumed to be homogeneous and in the class of stochastic diffusion processes:

$$\mathrm{d}D_t = b(D_t)\,\mathrm{d}t + \sigma\,\mathrm{d}w_t. \tag{18}$$

The problem is to identify the function $b(\cdot)$ and the number σ. We consider the empirical frequencies as a function of D, noting $f_r(D)$, where r

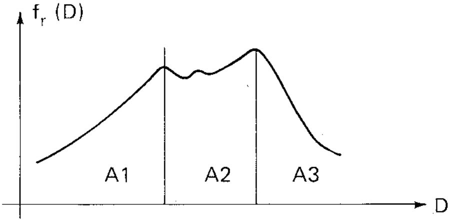

Figure 7.3

is the size of the same (24 × 365). We then obtain fig. 7.3, where we make a partition of the domain into three areas $\{A_1, A_2, A_3\}$.

We look for $b(D)$ as the function:

$$b(D) = \sum_{i=1}^{3} \theta^i \mathbf{1}_{A_i}(D), \tag{19}$$

where $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 if $x \notin A$, and θ^i, $i = 1, 2, 3$, and σ are the parameters to identify. The maximum likelihood estimator of θ_i is given by

$$\hat{\theta}_t^i = \frac{\int_0^t \mathbf{1}_{A_i}(D_t)\,\mathrm{d}D_t}{\int_0^t \mathbf{1}_{A_i}(D_t)\,\mathrm{d}t}, \tag{20}$$

$$\hat{b}_T(D) = \sum_{i=1}^{3} \hat{\theta}_t^i \mathbf{1}_{A_i}(D). \tag{21}$$

The estimator of σ^2 is then given by

$$\hat{\sigma}^2 = \sum_{i=1}^{r} \frac{\left[D_{t_{i+1}} - D_{t_i} - \hat{b}_T(D_{t_i})(t_{i+1} - t_i)\right]^2}{T}$$

Remark 7.3. The choice of the partition $\{A_i\}$ comes from the following remark:

$$\begin{cases} f_r(D)_{r\to\infty} \to p(D) \text{ solution of} \\ -\dfrac{\partial}{\partial D}(b(D)p) + \dfrac{\sigma^2}{2}\dfrac{\partial^2}{\partial D^2}p = 0, \\ \displaystyle\int p(D)\,\mathrm{d}s = 1. \end{cases} \tag{22}$$

The solution of (22) is

$$p(D) = C_1 \exp(-V(D)) \tag{23}$$

with

$$V(D) = \int_0^D 2\frac{b(D)}{\sigma^2}\, \mathrm{d}D, \tag{24}$$

and therefore $p(D)$ reaches its maximum at a value D^* where

$$b(D^*) = 0, \quad \text{with} \begin{cases} b(D) \leqslant 0, & \text{if } D \geqslant D^*, \\ b(D) \geqslant 0, & \text{if } D \leqslant D^*. \end{cases}$$

The shape of the partition has been chosen in such a way that the discontinuities on $\hat{b}$ are the values where b changes its sign.

4.2. *Identification of the long-run demand*

The data given are the means over one week of the demand. It can be seen that the observations remain around the annual mean. We have modelled the long-run demand by a linear diffusion process:

$$\mathrm{d}D_t = (\alpha_1 D_t + \alpha_2)\, \mathrm{d}t + \sigma\, \mathrm{d}w_t. \tag{25}$$

The maximum likelihood estimators are given by

$$\hat{\alpha}_1 = \frac{T\int_0^T D_t\, \mathrm{d}D_t - \left(\int_0^T \mathrm{d}D_t\right)\left(\int_0^T D_t\, \mathrm{d}t\right)}{T\int_0^T D^2\, \mathrm{d}t - \left(\int_0^T D_t\, \mathrm{d}t\right)^2}, \tag{26}$$

$$\hat{\alpha}_2 = \frac{\left(\int_0^T D_t\, \mathrm{d}D_t\right)\left(\int_0^T D_t\, \mathrm{d}t\right) - \left(\int_0^T \mathrm{d}D_t\right)\left(\int_0^T D_t^2\, \mathrm{d}t\right)}{\left(\int_0^T D_t\, \mathrm{d}t\right)^2 - T\int_0^T D^2\, \mathrm{d}t}. \tag{27}$$

Then, with the notation

$$\hat{b}_t(D) = \hat{\alpha}_1 D + \hat{\alpha}_2,$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{r}\left(D_{t_i+1} - D_{t_i} - \hat{b}_T(D_{t_i})(t_{i+1} - t_i)^2\right). \tag{28}$$

Remark 7.4. A good approximation for a discretized problem would be to consider the random variables D_t as independent with the stationary probability law

$$-\frac{\partial}{\partial D}\left[(\alpha_1 D + \alpha_2)p\right] + \tfrac{1}{2}\sigma^2\frac{\partial^2 p}{\partial D^2} = 0,$$

$$\int p \, \mathrm{d}D = 1.$$

4.3. *Identification of the short-run water supplies*

We assume that supplies can be modelled by a diffusion process with independent increments

$$A_t = \tilde{\beta}t + \tilde{\sigma}W_t, \tag{29}$$

where $\tilde{\beta}$ and $\tilde{\sigma}$ are assumed to be given by the local meteorological office.

4.4. *Identification of the long-run water supplies*

Supplies are modelled by

$$A_t = \int_0^T \beta(s)\,\mathrm{d}s + \sum_{s<t}(A_s - A_{s^-}). \tag{30}$$

We have to estimate the function $\beta(s)$ and the probability law of the jumps $\nu(t, \mathrm{d}u)\,\mathrm{d}t$. We assume that $s \to \beta(s)$ and $s \to \nu(s, \mathrm{d}u)$ are one year periodic functions (T = one year).

The estimator of β is given by

$$\hat{\beta}(t) = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{X}(t + \mathrm{i}T) \wedge U_M\right), \tag{31}$$

where n is the number of years observed.

We construct a partition $\{B_i\}$ of the set $[0, T] \times \{R^+ - \{0\}\}$, and the probability law $\nu(t, \mathrm{d}u)\,\mathrm{d}t$ is approximated by

$$\nu(t, \mathrm{d}u)\,\mathrm{d}t = \sum_i \theta_i \mathbf{1}_{B_i}(\mathrm{d}u, \mathrm{d}t)$$

and then the estimator of θ_i is

$$\hat{\theta}_i = \frac{\mathrm{card}\{(t, u) | u = A_t - A_{t^-}, (t, u) \in B_i\}}{\int_{B_i} \mathrm{d}u\,\mathrm{d}t}. \tag{32}$$

$\hat{\lambda}^- = n/\tau$, where n = number of breakdowns when the four thermal power plants of type 1 are working together, and τ = length of working time of the four thermal power plants of type 1 together. (33)

$\hat{\lambda}^+ = n'/\tau'$, where n' = total number of repairs when three thermal power plants of type 1 where working simultaneously, and τ' = time of working of three thermal power plants of type 1 when the fourth was not in working order. (34)

$\hat{\mu} = n''/\tau''$, where n'' = total number of breakdowns of thermal power plants of type 2, and $\tau'' = \int_0^T N_{t^-}\,\mathrm{d}t$ = total operating time of thermal power plants of type 2. (35)

5. Optimization of the management of the system of production

We will use discretization techniques to solve the problem; to this end we need to introduce the following notation:

h_T = discretization step length for time
h_D = discretization step length for demand
h_S = discretization step length for stock
i_T = point of discretization of time
i_D = point of discretization of demand
i_S = point of discretization of stock
$i_M \in \{3, 4\}$ = possible values of M
$i_N \in \{0, 1, 2, 3, 4\}$ = possible values of N
$P^D_{i_D, i_{D'}}$ = transition probability from the point i_D to the point $i_{D'}$
$P^{S,u}_{i_S, i_{S'}}$ = transition probability from the point i_S to $i_{S'}$
$P^M_{i_M, i_{M'}}$ = transition probability of the number of thermal power plants of type 1 in activity
$P^{N, N_1}_{i_N, i_{N'}}$ = transition probability of the number of thermal power plants of type 2 from i_N to $i_{N'}$ knowing that N_1 are wanted in activity
$V(i_t, i_D, i_S)$ or $(V(t, D, S)$ = dynamic programming optimal return function for the long-run problem

$\tilde{V}(t, D, S, M, N)$
or $\tilde{V}(i_t, i_D, i_S, i_M, i_N)$ = dynamic programming optimal return function for the short-run problem
n_T = number of discretization points in time
n_D = number of discretization points in D
n_S = number of discretization points in S
q = number of elements of the partition $\{B_i\}$ (see subsection 4.4).

5.1. Long-run optimization of the management of means of production

We use the dynamic programming method, although the methods developed in Breton–Falgarone (1973) and Falgarone–Lederer (1978) also give good results.

We note

$$V(t, D, S) = \min_{u, p} \mathrm{E}^{t, D, S} \int_t^T C(P_t)\, \mathrm{d}t, \tag{36}$$

where

$$P_t = D_t - \mathrm{E}(S_t, u_t). \tag{37}$$

V satisfies the following dynamic programming equation:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial D^2} + b \frac{\partial V}{\partial D} + \int_0^{S_M - S} [V(S + y) - V(s)]\nu(t, \mathrm{d}y)$$

$$+ (V(S_{\mathrm{M}}) - V(S))\nu(t, [S_{\mathrm{M}} - S, +\infty])$$

$$+ \min_u \left\{ (\beta - u)\frac{\partial V}{\partial S} + C(D - \mathrm{E}(S, u)) \right\} = 0$$

$$V(T, S, D) = 0. \tag{38}$$

Numerical solution

We use the probabilistic methods of discretization developed in Kushner (1977). The problem (38) is approached by the following Markov chain control problem.

Let P^D be the tridiagonal transition matrix $(n_D \times n_D)$:

$$P^D = \begin{bmatrix} 1 - \frac{b_0^+ h_t}{h_D} - \frac{1}{2}\sigma^2 \frac{h_t}{h_D^2} & \frac{b_0^+ h_t}{h_D} + \frac{1}{2}\sigma^2 \frac{h_t}{h_D^2} & 0 \\ \cdots & \cdots & \cdots \\ \frac{b_i^- h_t}{h_D} + \frac{1}{2}\sigma^2 \frac{h_t}{h_D^2} & 1 - \frac{|b_i h_t|}{h_D} - \sigma^2 \frac{h_t}{h_D^2} & \frac{b_i^+ h_t}{h_D} + \frac{1}{2}\sigma^2 \frac{h_t}{h_D^2} \\ \cdots & \cdots & \cdots \\ 0 & b_{n_D}^- \frac{h_t}{h_D} + \frac{1}{2}\sigma^2 \frac{h_t}{h_D^2} & 1 - b_{n_D}^- \frac{h_t}{h_D} - \frac{1}{2}\sigma^2 \frac{h_t}{h_D^2} \end{bmatrix}, \tag{39}$$

$$P^{S,u} = \bar{P}^{S,u} + \bar{P}^S, \tag{40}$$

with

$$\bar{P}^S = \begin{bmatrix} 1 - \sum_{i=1}^{q} \theta_i h_t h_S & \theta_1 h_t h_S & & \sum_{j=n_S}^{q} \theta_j h_t h_S \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 - \sum_{i=1}^{q} \theta_i h_t h_S & \theta_{i+1} h_t h_S \ldots & \sum_{j=n_S-i+1}^{q} \theta_j h_t h_S \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & 1 \end{bmatrix}, \tag{41}$$

where n_S is the number of points of discretization of S and q the number of elements of the partition of $\{B_i\}$ (defined in subsection 4.4).

$\bar{P}^{S,u}$ is the tridiagonal matrix

$$\bar{P}^{S,u} = \begin{bmatrix} -(\beta - u)_0^+ \frac{h_t}{h_S} & (\beta - u)_0^+ \frac{h_t}{h_S} & 0 \\ \cdots & \cdots & \cdots \\ (\beta - u)_i^- \frac{h_t}{h_S} & -|\beta - u|_i \frac{h_t}{h_S} & (\beta - u)_i^+ \frac{h_t}{h_S} \\ \cdots & \cdots & \cdots \\ 0 & (\beta - u)_{n_S}^- \frac{h_t}{h_S} & -(\beta - u)_{n_S}^- \frac{h_t}{h_S} \end{bmatrix}. \tag{42}$$

For the control problem of the Markov chain with states i_D, i_S, the dynamic programming equation is

$$V(i_t, i_D, i_S) = \min_{u_m < u < u_M} \Big\{ C_{i_D, i_S}(u) h_t$$

$$\times + \sum_{i'_D, i'_S} P^D_{i_D, i'_D} P^{S, u}_{i_S, i'_S} V(i_t + 1, i'_D, i'_S) \Big\}, \tag{43}$$

with

$$C_{i_D, i_S} = C(i_D h_D - \mathrm{E}(i_S h_S, u)). \tag{44}$$

5.2. Short-run optimization of the management of means of production

We denote

$$\tilde{V}(t, D, S, M, N) = \min \mathrm{E}^{t, D, S, M, N}$$
$$\times \int_t^T \{\rho_1 P_1 + \rho_2 P_2 + \rho_3 (D - P_1 - P_2 - \mathrm{E}(S, u))^+\} \, \mathrm{d}t$$
$$+ \sum_{t \leqslant s \leqslant T} k(N_1 - N_{s^-} + \Delta N_s^2)^+ + V(T, D_T, S_T). \tag{45}$$

The control variables are u, P_1, P_2, and N^1.

V is the solution of the following dynamic programming equation, also called QVI (quasivariational inequalities), due to Bensoussan–Lions (1978) and Robin (1977):

$$\min \Bigg\{ \frac{\partial \tilde{V}}{\partial t} + b \frac{\partial \tilde{V}}{\partial D} + \tfrac{1}{2} \sigma^2 \frac{\partial^2 \tilde{V}}{\partial D^2} + \tfrac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 \tilde{V}}{\partial S^2} + \lambda^+ (\tilde{V}(M+1) - \tilde{V}(M))$$

$$+ \lambda^- (\tilde{V}(M-1) - \tilde{V}(M)) + N\mu (\tilde{V}(N-1) - V(N))$$

$$+ \min_{u, P^1, P^2} \Bigg[\rho_1 P_1 + \rho_2 P_2 + \rho_3 (D - P_1 - P_2 - \mathrm{E}(S, u))^+$$

$$+ \frac{\partial \tilde{V}}{\partial S} (\tilde{\beta} - u) \Bigg];$$

$$\min_{N_1} \left[\tilde{V}(N_1) + k(N_1 - N)^+ - \tilde{V}(N) \right] \Bigg\} = 0. \tag{46}$$

We define the following transition matrix:

$$P^{S,u} = \begin{bmatrix} 1-(\tilde\beta-u)_0^+\frac{h_t}{h_S}-\frac{1}{2}\tilde\sigma^2\frac{h_t}{h_S^2} & \frac{(\tilde\beta-u)^+h_t}{h_S}+\frac{1}{2}\tilde\sigma^2\frac{h_t}{h_S^2} & 0 & \\ \cdots & \cdots & \cdots & \cdots \\ 0(\tilde\beta-u)_i^-\frac{h_t}{h_S}+\frac{1}{2}\tilde\sigma^2\frac{h_t}{h_S^2} & 1-(\tilde\beta-u)_i\frac{h_t}{h_S}-\tilde\sigma^2\frac{h_t}{h_S^2} & (\tilde\beta-u)_i^+\frac{h_t}{h_S}+\frac{1}{2}\tilde\sigma^2\frac{h_t}{h_S^2} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & (\tilde\beta-u)_{nS}^-\frac{h_t}{h_S}+\frac{1}{2}\tilde\sigma^2\frac{h_t}{h_S^2} & 1-(\tilde\beta-u)_{nS}^-\frac{h_t}{h_S}-\frac{1}{2}\tilde\sigma^2\frac{h_t}{h_S^2} & \end{bmatrix}, \tag{47}$$

$$P^M = \begin{matrix} & 3 & 4 \\ 3 & 1-\lambda^+h_t & \lambda^+h_t \\ 4 & \lambda^-h_t & 1-\lambda^-h_t \end{matrix}, \tag{48}$$

$$P^{N,N_1} = \begin{matrix} & N_1-1 & N_1 & \\ \Big[\,0\ldots & N\mu h_t & 1-N\mu h_t & \ldots 0\,\Big] \\ \Big[\,0\ldots & N\mu h_t & 1-N\mu h_t & \ldots 0\,\Big] \end{matrix}, \tag{49}$$

P_D is as in (39), but with the short-run coefficients b and σ. (50)

The problem of controlling the Markov chain with states i_D, i_S, i_M, i_N and the preceding transition probability matrix leads to solve the following dynamic programming equation:

$$V(i_t, i_D, i_S, i_M, i_N)$$

$$= \min_{\substack{0 \leqslant N_1 \leqslant 4 \\ u_m \leqslant u \leqslant u_M \\ i_M P_m^1 \leqslant P_1 \leqslant i_M P_M^1 \\ i_N P_m^2 \leqslant P_2 \leqslant i_N P_M^2}} \Bigg\{ \rho_1 P_1 + \rho_2 P_2 + \rho_3 [D - P_1 - P_2 - \mathrm{E}(i_S h_S, u)]^+ h_t$$

$$+ k(N_1 - i_N)^+ + \sum_{\substack{i'_D \\ i'_S \\ i'_M \\ i'_N}} P^D_{i_D, i'_D} P^{S,u}_{i_S, i'_S} P^M_{i_M, i'_M} P^{N,N_1}_{i_N, i'_N}$$

$$\times V(i_t + 1, i'_D, i'_S, i'_M, i'_N) \Bigg\}. \tag{51}$$

5.3. Study of the particular case $E(S, u) = (1 - e^{-\beta u})f(s)$

In this particular case we can calculate the value of the minimum appearing in (51). We denote

$$\phi(S, u) = \min_{\substack{MP_m^1 < P_1 < MP_M^1 \\ NP_m^2 < P_2 < NP_M^2}} \rho_1 P_1 + \rho_2 P_2 + \rho_3 (D - P_1 - P_2 - \mathrm{E}(S, u))^+ , \tag{52}$$

$$\phi(S, u) = \begin{cases} \rho_1 MP_m^1 + N\rho_2 P_m^2, & \text{if } d - \mathrm{E}(S, u) < P_{\min}, \\ (\rho_2 - \rho_1) NP_m^2 + \rho_1 (d - \mathrm{E}(S, u)), & \text{if } P_{\min} < d - \mathrm{E}(S, u) < P_{\mathrm{moy}}, \\ (\rho_1 - \rho_2) MP_M^1 + \rho_2 (d - \mathrm{E}(S, u)), & \text{if } P_{\mathrm{moy}} < d - \mathrm{E}(S, u) < P_{\max}, \\ d - \mathrm{E}(S, u) - (1 - \rho_1) MP_M^1 - (1 - \rho_2) NP_M^2, & \text{if } P_{\max} < d - \mathrm{E}(S, u). \end{cases} \tag{53}$$

The unit price has been fixed at $\rho_3 = 1$.

We recall the notations

$$P_{\min} = MP_m^1 + NP_m^2, \tag{54}$$

$$P_{\mathrm{moy}} = MP_M^1 + NP_m^2, \tag{55}$$

$$P_{\max} = MP_M^1 + NP_M^2. \tag{56}$$

The solution of the equation

$$d - \mathrm{E}(S, u) = p \tag{57}$$

with

$$\mathrm{E}(S, u) = f(s)(1 - \mathrm{e}^{-\beta u}), \tag{58}$$

can be written

$$u = -\frac{1}{\beta} \log\left(1 - \frac{d - p}{f(s)}\right), \tag{59}$$

and so with the notations

$$u_{P_{\min}} = -\frac{1}{\beta} \log\left(1 - \frac{d - P_{\min}}{f(S)}\right), \tag{60}$$

$$u_{P_{\mathrm{moy}}} = -\frac{1}{\beta} \log\left(1 - \frac{d - P_{\mathrm{moy}}}{f(S)}\right), \tag{61}$$

$$u_{P_{\max}} = -\frac{1}{\beta} \log\left(1 - \frac{d - P_{\max}}{f(S)}\right), \tag{62}$$

eq. (53) becomes

$$\phi(S, u) = \begin{cases} \rho_1 MP_{\mathrm{m}}^1 + \rho_2 NP_{\mathrm{m}}^2, & u_{\mathrm{M}} \geqslant u \geqslant u_{P_{\min}}, \\ (\rho_2 - \rho_1) NP_{\mathrm{m}}^2 + \rho_1(d - E(S, u)), & u_{P_{\min}} \geqslant u \geqslant u_{P_{\mathrm{moy}}}, \\ (\rho_1 - \rho_2) MP_{\mathrm{M}}^1 + \rho_2(d - E), & u_{P_{\mathrm{moy}}} \geqslant u \geqslant u_{P_{\max}}, \\ (d - E) - (1 - \rho_1) MP_{\mathrm{M}}^1 - (1 - \rho_2) NP_{\mathrm{M}}^2, & u_{P_{\max}} \geqslant u \geqslant u_{\mathrm{m}}, \end{cases} \tag{63}$$

eq. (51) can be written

$$\tilde{V}(i_t, i_D, i_S, i_M, i_N) = \min_{\substack{u_{\mathrm{m}} \leqslant u \leqslant u_{\mathrm{M}} \\ 0 \leqslant N_1 \leqslant 4}} \left\{ \phi(S, u) + \sum_{i'_S} P_{i_S, i'_S}^{S, u} v(i'_s) \right\}. \tag{64}$$

With the notation

$$v(i'_S) = \sum_{\substack{i'_D \\ i'_M \\ i'_N}} \tilde{V}(i_t + 1, i'_D, i'_S, i'_M, i'_N) P_{i_D, i'_D}^{D} P_{i_M, i'_M}^{M} P_{i_N, i'_N}^{N, N_1}, \tag{65}$$

eq. (57) is rewritten

$$\begin{aligned} \tilde{V}(t, i_D, i_S, i_M, i_N) = \min_{0 \leqslant N_1 \leqslant 4} \Bigg\{ & \left(1 - \tilde{\sigma}^2 \frac{h_t}{h_S^2}\right) v(i_S) + \tfrac{1}{2} \tilde{\sigma}^2 \frac{h_t}{h_S^2} v(i_S + 1) \\ & + \tfrac{1}{2} \tilde{\sigma}^2 \frac{h_t}{h_S^2} v(i_S - 1) + h_t \min_{u_{\mathrm{m}} \leqslant u \leqslant u_{\mathrm{M}}} \\ & \times \Bigg[(\tilde{\beta} - u)_{i_S}^{+} \frac{v(i_S + 1) - v(i_S)}{h_S} \\ & - (\tilde{\beta} - u)_{i_S}^{-} \frac{v(i_S) - v(i_S - 1)}{h_S} + \phi(i_S, u) \Bigg] \Bigg\}. \end{aligned} \tag{66}$$

Then we have to calculate

$$\begin{aligned} \min_{u_{\mathrm{m}} \leqslant u \leqslant u_{\mathrm{M}}} \Bigg\{ & (\tilde{\beta} - u)^{+} \frac{v(i_S + 1) - v(i_S)}{h_S} \\ & - (\beta - u)^{-} \frac{v(i_S) - v(i_S - 1)}{h_S} + \phi(i_S, u) \Bigg\}. \end{aligned} \tag{67}$$

To solve (60), it is sufficient to solve the equation

$$\frac{\partial \phi}{\partial u}(S, u) = \alpha, \quad \text{with unknown } u, S \text{ fixed.} \tag{68}$$

By using (63) the solution of (68), illustrated in fig. 7.4, is given by

$$u(\alpha) = \begin{cases} u_{p_{\min}} \wedge u_{\max}, \\ \quad \text{if } \alpha \geqslant -\rho_1 \tilde{f}(S)\exp(-\beta u_{p_{\min}}), \\ -\dfrac{1}{\beta}\log\dfrac{\alpha}{\rho_1 \tilde{f}(S)}, \\ \quad \text{if } -\rho_1 \tilde{f}\exp(-\beta u_{p_{\min}}) \geqslant \alpha \geqslant -\rho_1 \tilde{f}(S)\exp(-\beta u_{p_{\text{moy}}}), \\ u_{p_{\text{moy}}}, \\ \quad \text{if } -\rho_1 \tilde{f}\exp(-\beta u_{p_{\text{moy}}}) \geqslant \alpha \geqslant -\rho_2 \tilde{f}(S)\exp(-\beta u_{p_{\text{moy}}}), \\ -\dfrac{1}{\beta}\log\dfrac{\alpha}{\rho_2 \tilde{f}(S)}, \\ \quad \text{if } -\rho_2 \tilde{f}\exp(-\beta u_{p_{\text{moy}}}) \geqslant \alpha \geqslant -\rho_2 \tilde{f}(S)\exp(-\beta u_{p_{\max}}), \\ u_{p_{\max}}, \\ \quad \text{if } -\rho_2 \tilde{f}\exp(-\beta u_{p_{\max}}) \geqslant \alpha \geqslant -\tilde{f}(S)\exp(-\beta u_{p_{\max}}), \\ -\dfrac{1}{\beta}\log\dfrac{\alpha}{\tilde{f}(S)} \vee 0, \\ \quad \text{if } -\tilde{f}\exp(-\beta u_{p_{\max}}) \geqslant \alpha, \end{cases} \tag{69}$$

with $\tilde{f}(s) = \beta f(s)$ and $x \vee y = \max(x, y)$.

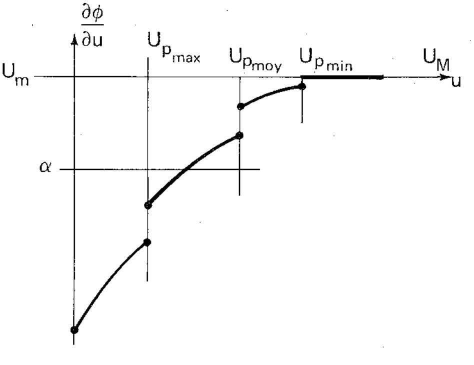

Figure 7.4

We compute then

$$u^1 = u\left(\frac{v(i_S+1)-v(i_S)}{h_s}\right), \tag{70}$$

$$u^2 = u\left(\frac{v(i_S)-v(i_S-1)}{h_s}\right). \tag{71}$$

u^* optimal is given by

$$u^* = \begin{cases} u_1, & u_1 \leqslant \tilde{\beta}, \quad u_2 \leqslant \tilde{\beta}, \\ \tilde{\beta}, & u_1 \geqslant \tilde{\beta}, \quad u_2 \leqslant \tilde{\beta}, \\ u_2, & u_1 \geqslant \tilde{\beta}, \quad u_2 \geqslant \tilde{\beta}, \\ \arg\min_{u_1, u_2} g(u), & u_1 \leqslant \tilde{\beta}, \quad u_2 \geqslant \tilde{\beta}, \end{cases} \tag{72}$$

with

$$g(u) = (\tilde{\beta}-u)^+ \frac{v(i_S+1)-v(i_S)}{h_s} - (\tilde{\beta}-u)^- \frac{v(i_S)-v(i_S-1)}{h_s} + \phi(S, u).$$

It only remains to provide an algorithm to determine N^1. We use complete enumeration here since $N_1 \in \{0, 1, \ldots, 4\}$ once the following monotonicity properties have been noted:

N_1 increases when i_D increases;
N_1 increases when i_S decreases;
N_1 increases when i_M decreases; and
N_1 increases when i_N decreases.

6. Numerical results

We now give realistic data for the situation in New Caledonia before 1977. For these data we give the numerical results that we have obtained.

6.1. Long-run problem

$T = 1$ (year)
$S_M = 1$
$S_m = 0$

$$\begin{aligned}
u_M &= 68 \times 365 \times 24/(1.2 \times 10^5)\\
u_m &= 0\\
D_t &= \text{demand (MW)}/Q\\
Q &= 4 \times (16 + 37.75) + 68 + 21 = 305\\
P_{min} &= 0\\
P_{moy} &= 0.9 \times 4 \times 16/305\\
P_{max} &= 0.9 \times 4(16 + 37.75)/305.
\end{aligned}$$

The mean availability is 0.9.

$$\begin{aligned}
\rho_1 &= 2.18/3.475\\
\rho_2 &= 2.60/3.475\\
\rho_3 &= 1
\end{aligned}$$

$$\mathrm{E}(S, u) = 17(14/16 + \log 31.5\ S)/16 \times \log 5.6)\exp\left(1 - \frac{4 \times u}{305 \times u_M}\right).$$

A realization of the demand is given by fig. 7.5, and a realization of the water supply is given by fig. 7.6.

The following discretization parameters were used:

$$h_t = \frac{3}{365}; \qquad h_s = 0.1,$$

with points of discretization of the demand as follows:

$$(120, 140, \ldots, 250)/305.$$

The demand parameters were estimated to be

$$\hat{b}(D) = -420 \times (D - 293/420),$$
$$\hat{\sigma}^2 = 0.015.$$

with estimates of water supply parameters given in figs. 7.7 and 7.8.

The optimal strategy is illustrated in figs. 7.9 and 7.10.

6.2. Numerical results for the short-run problem

$$\begin{aligned}
T &= 1\\
S_M &= 1\\
S_m &= 0\\
U_M &= 68 \times 365 \times 24/(1.2 \times 10^5)\\
U_m &= 0\\
D(t) &= \frac{\text{demand (MW)}}{Q}
\end{aligned}$$

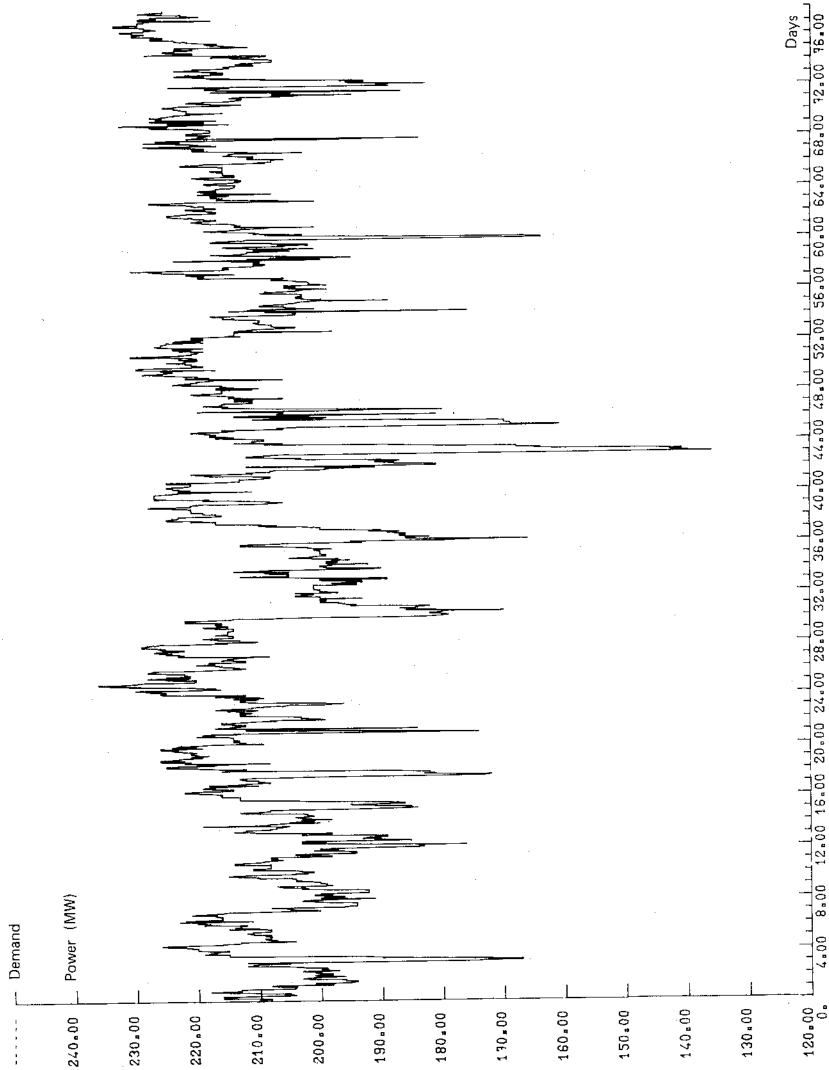
Demand
Power (MW)
Days
240.00
230.00
220.00
210.00
200.00
190.00
180.00
170.00
160.00
150.00
140.00
130.00
120.00
0.
4.00
8.00
12.00
16.00
20.00
24.00
28.00
32.00
36.00
40.00
44.00
48.00
52.00
56.00
60.00
64.00
68.00
72.00
76.00

Figure 7.5

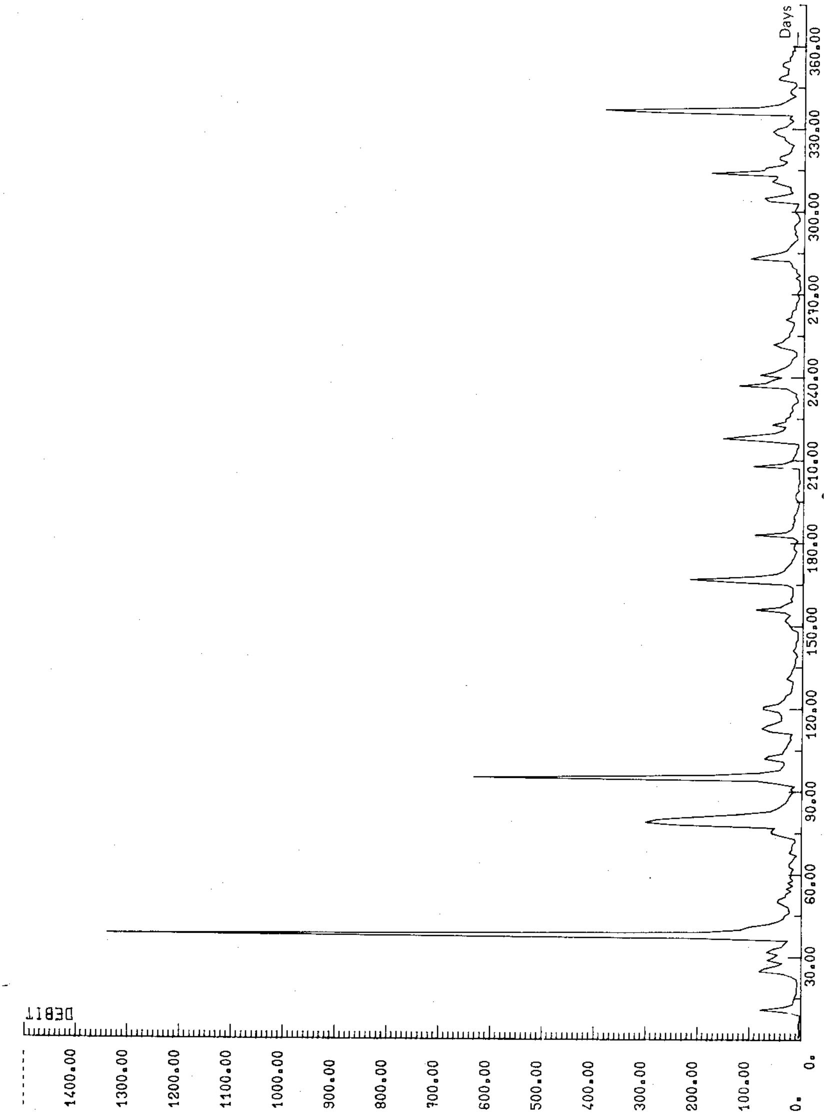
DEBIT
0.
100.00
200.00
300.00
400.00
500.00
600.00
700.00
800.00
900.00
1000.00
1100.00
1200.00
1300.00
1400.00
0.
30.00
60.00
90.00
120.00
150.00
180.00
210.00
240.00
270.00
300.00
330.00
360.00
Days

Figure 7.6

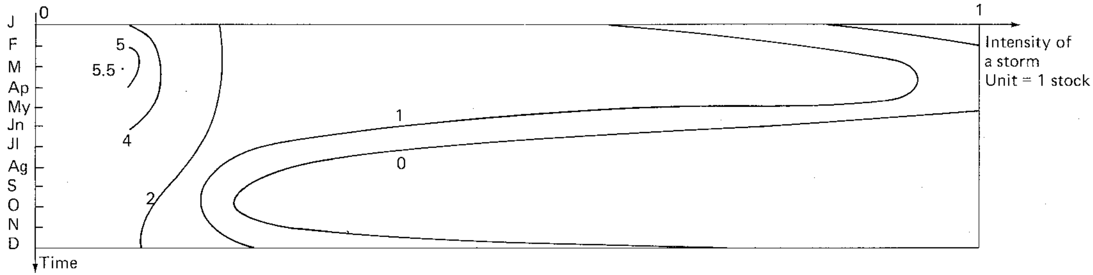

Figure 7.7. Smoothed curve with same frequencies as a function of time and their intensity, $\hat{\theta}(t, u)$.

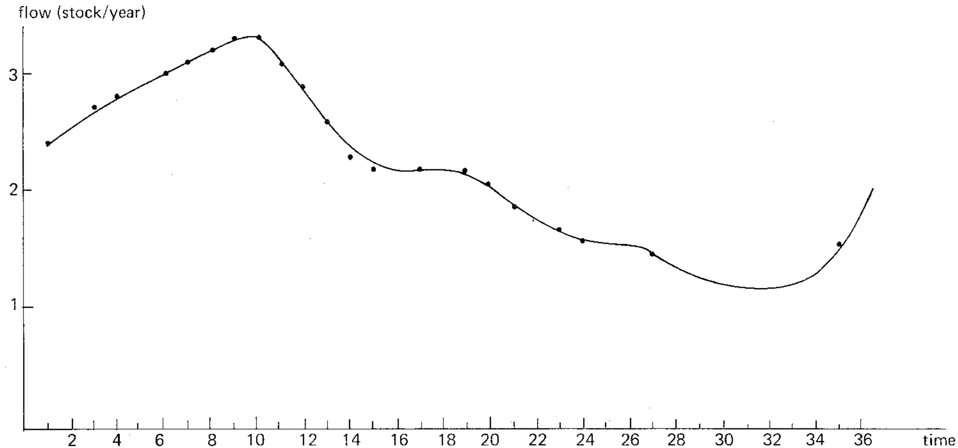

Figure 7.8. Mean continuous part of water inputs, $\hat{\beta}(t)$.

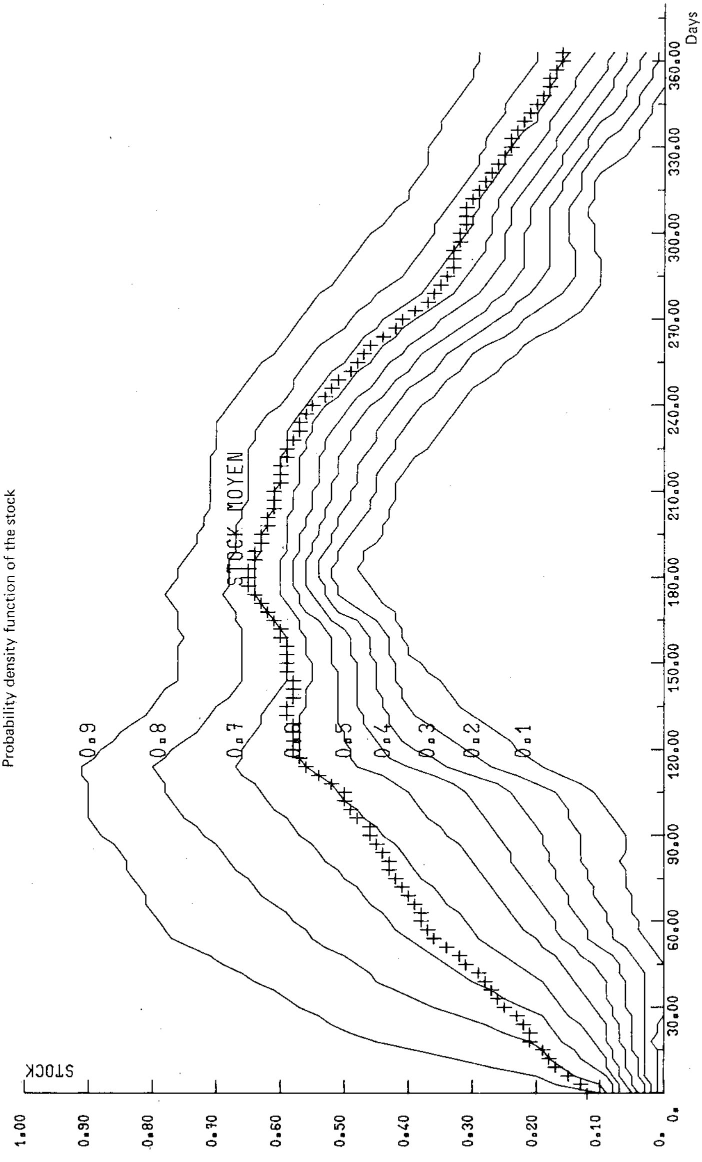

Figure 7.9. Probability density function of the stock.

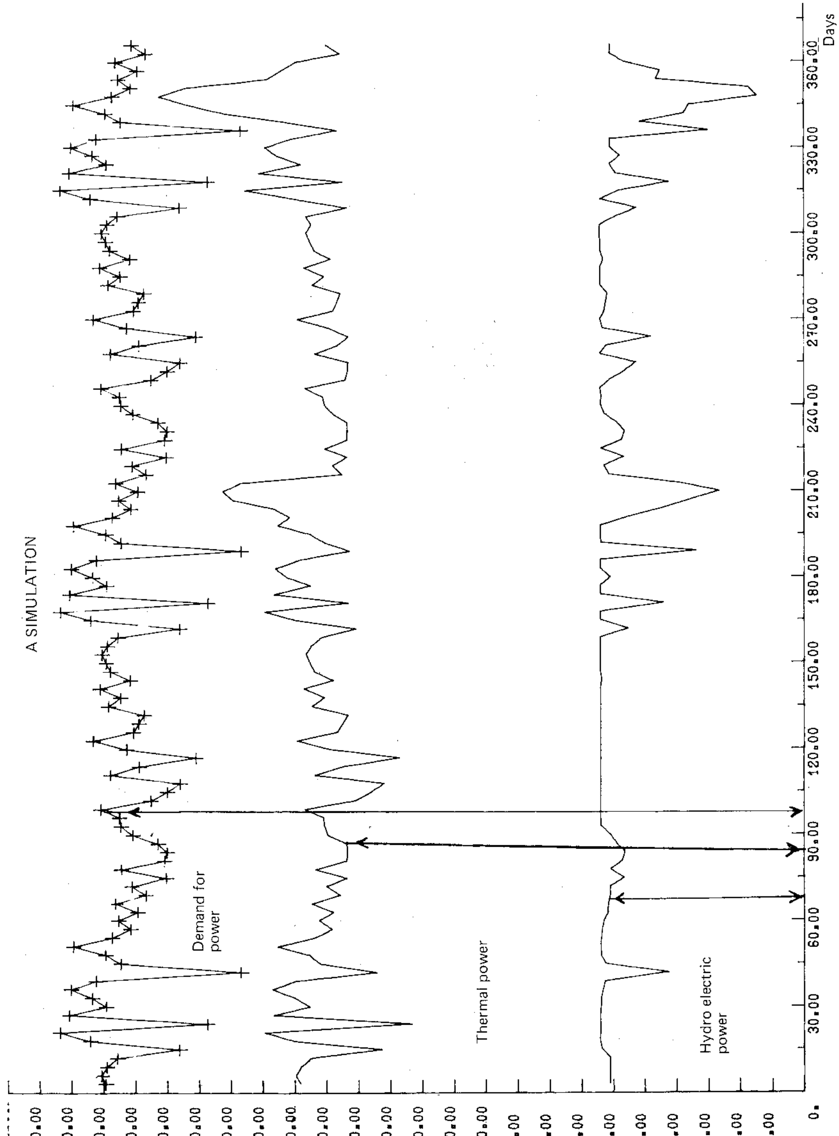

Figure 7.10. Demand for power: a simulation.

$$
\begin{aligned}
Q &= 305 \text{ MW} \\
P_{\mathrm{m}}^{1} &= 18/Q \\
P_{\mathrm{M}}^{1} &= 37.75/Q \\
P_{\mathrm{m}}^{2} &= 6/Q \\
P_{\mathrm{M}}^{2} &= 16/Q \\
\rho_1 &= 2.18/3.475 \\
\rho_2 &= 2.60/3.475 \\
\rho_3 &= 1 \\
k &= 2 \times 16 \times \rho_2/(Q \times 365 \times 24) \\
\mathrm{E}(S, u) &= 17 \times (14/16 + \log(31.5S)|16 \times \log 5.6)\exp\left(1 - \frac{4 \times U}{305 \times U_{\mathrm{M}}}\right) \\
h_t &= 2/(24 \times 365) \\
h_s &= 0.1.
\end{aligned}
$$

Points of discretization of the demand:

$$(120, 140, 160, 180, 195, 205, 215, 225, 235, 250)/305.$$

Probability law of the demand

$b(D)$ for $D \in \left[D_{i_D}, D_{i_D+1}\right]$

$(17, 17, 17, 17, 17, 17, -10, -25, -25)$

$\sigma^2(D) = (3.6, 3.6, 3.6, 3.6, 3.6, 3.6, 1.3, 2.4, 2.4).$

We have supposed σ piecewise constant instead of constant as indicated above.

Probability law of breakdowns

$$
\begin{aligned}
\lambda^+ &= 365 \\
\lambda^- &= 28 \\
\mu &= 29.
\end{aligned}
$$

Probability law of water supply

$$\tilde{\beta} = 2$$

$$(\tilde{\sigma})^2 = 0.2 \times 2/\,(365 \times 24).$$

Optimal feedbacks strategies are given in figs. 7.11 and 7.12.

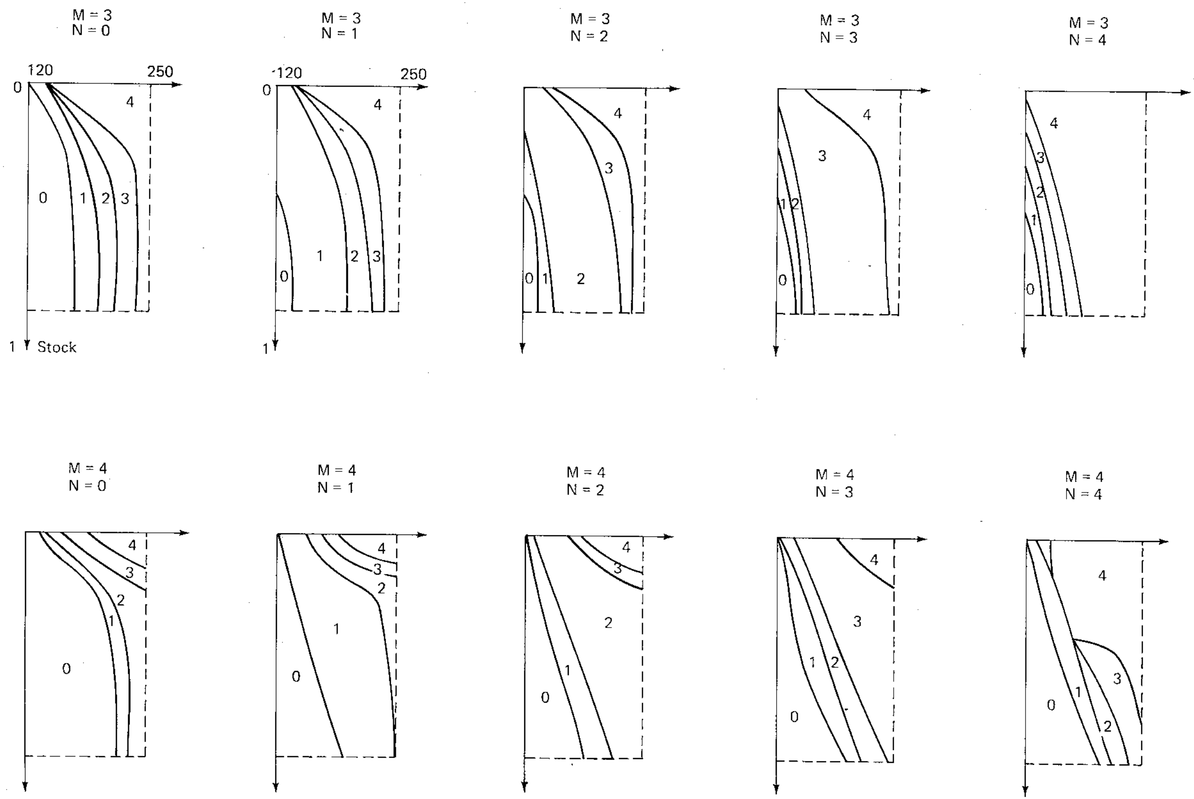

Figure 7.11. Start-up of small thermal power plants. Number of small power plants wanted in action as a function of the demand D, the stock S, and the number of thermal plants (M, N) in action.

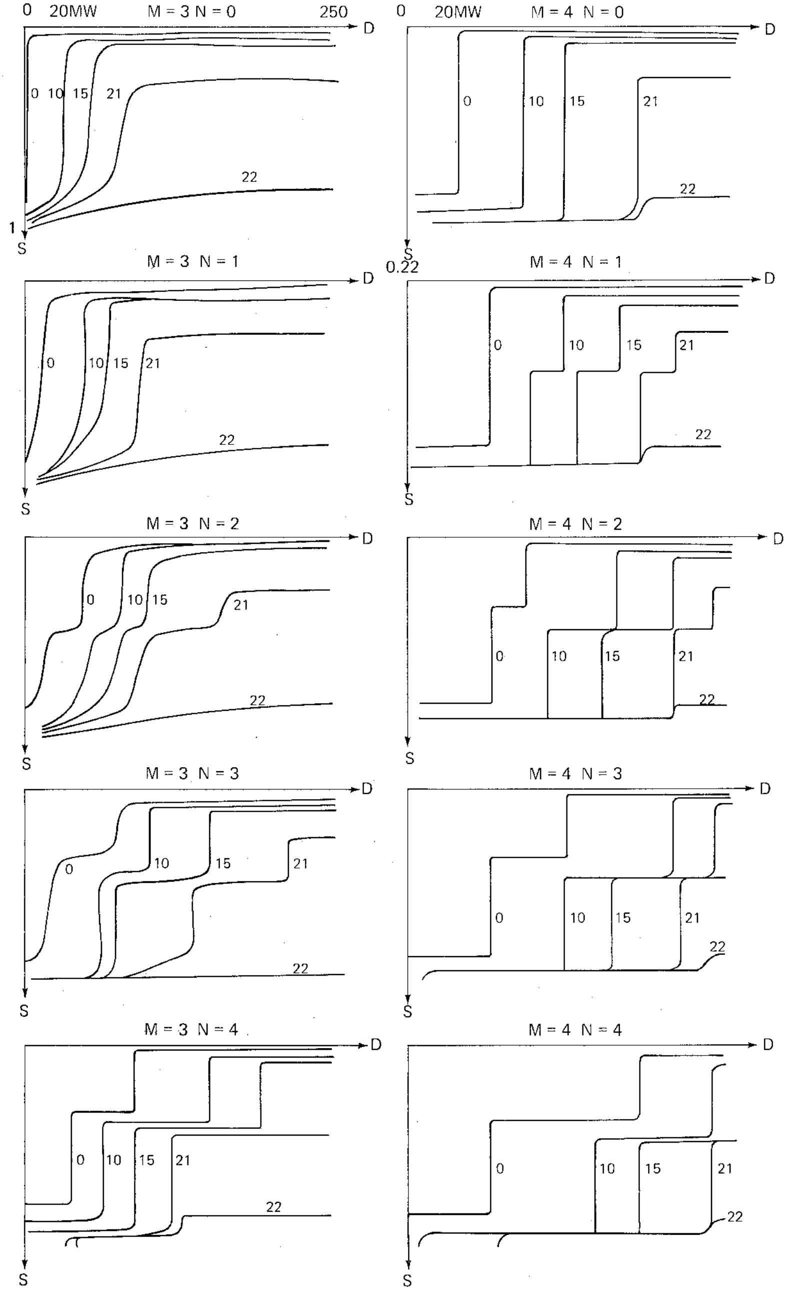

Figure 7.12. Quantity of water turbined as a function of the stock (*S*, the demand *D*, and the number of power plants in action. The stock is expressed as a percentage of the maximal storage capacity. The quantity turbined is expressed as a percentage of the maximal equipped power.

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