Degree one homogeneous minplus dynamic systems and traffic applications : Part I

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Abstract. We show that car traffic on a town can be modeled using a Petri net extension where arcs have negative weights. The corresponding minplus dynamics is not linear but homogeneous of degree one. Possibly depending on the initial condition, homogeneous of degree 1 minplus systems may be periodic or have a chaotic behavior (to which corresponds a constant throughput) or may explode exponentially. In traffic systems, when this constant throughput exists it has the interpretation of the average car speed. In this first part we recall the derivation of the 1-homogeneous dynamics of traffic system and show that may exist such systems with chaotic behavior having a constant throughput.

1. Introduction

The traffic on a road has been studied with different points of view at macroscopical level for example :

- The Lighthill-Whitham-Richards Model [**6**] is the more standard one
$$\begin{cases}\partial_t\rho+\partial_x q=0\ ,\\ q=f(\rho),\end{cases}$$
where $q(x,t)=$ denotes the flow at time t and position x on the road, $\rho(x,t)=$ denotes density, f is a function given, called the fundamental traffic law. It plays for traffic the role of the perfect gas law for the fluid dynamics.
- The kinetic model (Prigogine-Herman [**7**]) gives the evolution of the density of particles $\rho(t,x,v)$ as a function of t,x and v the speed of particle
$$\partial_t\rho+v\partial_x\rho=C(\rho,\rho)\ ,$$
where $C(\rho,\rho)$ is an interacting term in general quadratic in ρ.

The second model is more costly in term of computation time and therefore not used in practice. The first one suppose the knowledge of the function f. This function has been studied experimentally or theoretically using simple microscopic model. Here, we will recall a way to derive a good approximation of this law from a simple minplus linear system based on a Petri net.

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The main purpose of this paper is to generalize this fundamental law to the 2D cases where roads have crossings. The original minplus linear model on a unique road cannot be generalized easily in term of Petri nets. We have proposed in a previous paper a way to solve the difficulty by using Petri net with negative weights. The dynamics of these Petri net can be written easily but are not anymore linear in minplus algebra but are homogeneous of degree 1. We recall here the derivation of these 1-homogeneous dynamics.

In the first part of this paper we show that we can compute the eigenvalues for these 1-homogeneous system but that chaotic dynamics may appear. In the second part we discuss the phases appearing in the fundamental diagram, obtained numerically, and describe new situations where we can prove that the system is periodic.

2. Traffic on a circular road

Let us recall the simplest model to derive the fundamental traffic law on a single road. The simplest way is to study the stationary regime on a circular road with a given number of vehicles and then to consider that this stationary regime is reached locally when the density is given on a standard road. We present two way to obtain this law : – by logical deduction from an exclusion process, – by computing the eigenvalue of a minplus system derived from a simple Petri net describing the road with the vehicles.

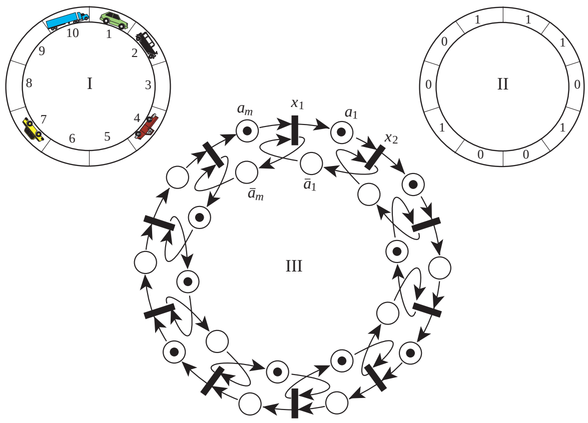

FIGURE 1. A circular road.

2.1. Exclusion process modeling. Following [**3**] we can consider the dynamic system defined by the rule $10 \to 01$ apply to a binary word describing the car positions on a road cut in section (each bit representing a section 1 meaning occupied and 0 meaning free see II in Figure 1). Let us take an example :

$$m_1 = 1101001001, \quad m_2 = 1010100101, \quad m_3 = 0101010011,$$
$$m_4 = 1010101010, \quad m_5 = 0101010101,$$

Let us define : – the *density* ρ the number of vehicles n divided by number of places m : $\rho = n/m$, – the flow $q(t)$ at time t the vehicle number going one step forward at time t divided by the place number. Then the *fundamental traffic law* gives the relation between $q(t)$ and d.

If $\rho \leq 1/2$, after a transient period, all the vehicle groups split off, and then all the vehicles can move forward without other vehicles in the way, and we have :

$$q(t) = q = n/m = d \ .$$

If $\rho \geq 1/2$, the free place groups split off after a finite time and move backward without other free place in the way. Then $m - n$ vehicles move forward and we have

$$q(t) = q = (m-n)/m = 1 - d \ .$$

Therefore :

$$\exists T : \ \forall t \geq T \quad q(t) = q = \begin{cases} \rho & \text{if } \ \rho \leq 1/2 \ , \\ 1-\rho & \text{if } \ \rho \geq 1/2 \ . \end{cases}$$

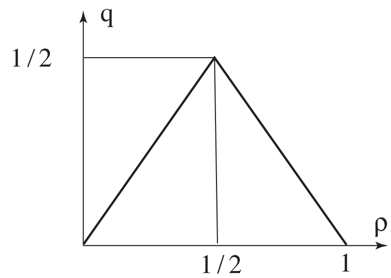

FIGURE 2. The fundamental traffic law.

2.2. Event Graph modeling. Consider the Petri net given in III of Figure 1 which describes in a different way the same dynamics. In fact this Petri net is an event graph and therefore its dynamics is linear in minplus algebra. The vehicle number entered in the place i before time k is denoted x_i^k. The initial vehicle position is given by booleans a_i with takes the value 1 when the cell contains a vehicle and 0 otherwise.

We use the notation $\bar{a} = 1 - a$, then the dynamics is given by :

$$x_i^{k+1} = \min\{a_{i-1} + x_{i-1}^k, \bar{a}_i + x_{i+1}^k\} \ ,$$

which can be written linearly in minplus algebra :

$$x_i^{k+1} = a_{i-1}x_{i-1}^k \oplus \bar{a}_i x_{i+1}^k \ .$$

This event graph has three kinds of elementary circuits : – the outside circuit with average mean n/m, – the inside circuit with average mean $(m-n)/m$, – the circuits corresponding to make some step forward and coming back, with average mean $1/2$, Therefore its eigenvalue is

$$q = \min(n/m, (m-n)/m, 1/2) = \min(\rho, 1-\rho) \ ,$$

which gives the average speed as a function of the car density.

3. 2D traffic

Let us generalize the second approach to derive the fundamental diagram to a regular town describe in Figure 3. The complete town can be modeled

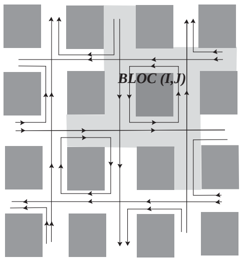

FIGURE 3. A town.

as a set of subsystems corresponding to a unique crossing and two adjacent roads. To write the dynamics of the town we have first to give the Petri net describing a crossing.

A first trial is to consider the Petri net given in Figure 4. This Petri net is not anymore an event graph but following L. Libeaut[**5**] it is possible to write the nonlinear implicit minplus equation describing a general Petri net. In the case where the multipliers are all equal to one it is :

$$\min_{p \in x^{in}} \left[a_p + \sum_{x' \in p^{in}} x'(k-1) - \sum_{x'' \in p^{out}} x''(k) \right] = 0, \ \forall x \ , \forall k. \tag{1}$$

where $x(k)$ denotes the firing number of transition x and p a place of the Petri Net. But these equations does not determine completely the dynamics

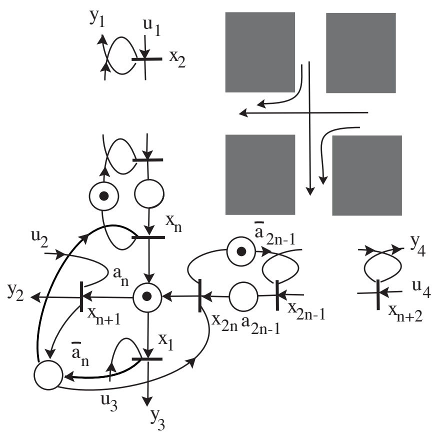

FIGURE 4. A simplified crossing.

since the Cauchy problem has not a unique solution. Indeed : – at place a_n we may have a *routing policy* giving the proportion of cars going towards y_2 and the proportion going towards y_3 (which is not described by the Petri net 4) – at place $\bar{a}_n$ we may follow *the first arrived the first served* rule with the right priority if two cars arrive simultaneously at the crossing (which is also not described by the Petri net 4).

Precising the dynamics of Petri net in such way that the trajectories are uniquely defined corresponds to give another Petri net having only one arc leaving each place. Let us discuss more precisely these points on a simple system given in the first picture of Figure 5.

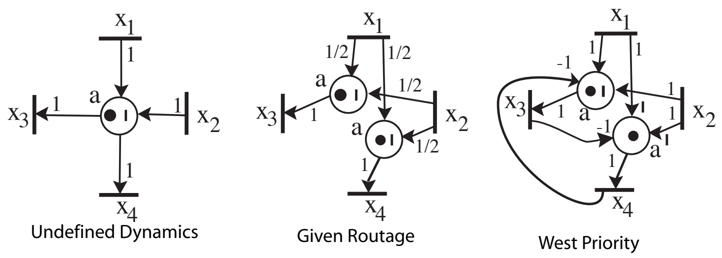

FIGURE 5. Dynamic Completion.

The incomplete dynamics of this system can be written in minplus algebra $x_4^n x_3^n = a x_1^{n-1} x_2^{n-1}$. Clearly x_4 and x_3 are not defined uniquely. We can complete the dynamics, for example, in the two following ways useful for the traffic application : – by precising the routing policy

$$x_4^n = x_3^n = \sqrt{a x_1^{n-1} x_2^{n-1}}$$

– by choosing a priority rule

$$\begin{cases} x_3^n = a x_1^{n-1} x_2^{n-1} / x_4^{n-1} \\ x_4^n = a x_1^{n-1} x_2^{n-1} / x_3^n . \end{cases}$$

In the two cases we obtain a *degree one homogeneous minplus* system.

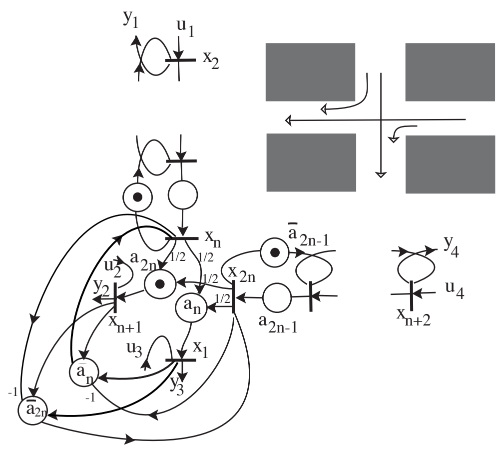

FIGURE 6. A Complete Crossing.

This method can be applied to the crossing and we obtain a Petri net with negative weights which has only one arc leaving each place (that we call deterministic Petri net) see Figure 6.

Neglecting the roundings the system can be written with minplus notations :

$$\begin{cases} x_i/\delta = a_{i-1}x_{i-1} \oplus \bar{a}_i x_{i+1}, \\ x_n/\delta = \bar{a}_n x_1 x_{n+1}/x_{2n} \oplus a_{n-1}x_{n-1} \ , \\ x_{2n}/\delta = \bar{a}_{2n} x_1 x_{n+1}/(x_n/\delta) \oplus a_{2n-1}x_{2n-1} \ , \\ x_1/\delta = a_n\sqrt{x_n x_{2n}} \oplus \bar{a}_1 x_2 \ , \\ x_{n+1}/\delta = a_{2n}\sqrt{x_n x_{2n}} \oplus \bar{a}_{n+1}x_{n+2} \ , \end{cases}$$

where δ denotes the forward shifting operator acting on sequences. It is a general degree 1 homogeneous minplus system.

Simulation of this system starting from 0 shows that

$$\lim_k x_i^k/k = \lambda, \ \ \forall i \ .$$

The constant λ has the interpretation of the average speed. The fundamental diagram gives the relation between the average speed and the vehicle density of the system. In Figure 7 we give this law in the cases of two circular roads with one crossing for different relative size of the two roads. We see that three phases appear on each fundamental diagram. These phases will be discussed in the second part of this paper. The experimental existence

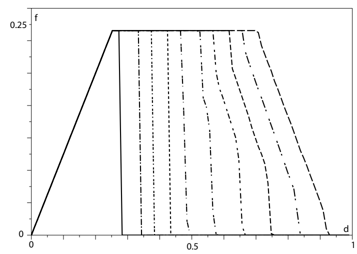

FIGURE 7. 2D-traffic fundamental diagrams.

of this λ motivates the study of the eigenvalue of 1-homogeneous minplus system.

4. Eigenvalues of 1-homogeneous minplus systems

The eigenvalue problem for 1-homogeneous system $f : \mathbb{R}^n_{\min} \mapsto \mathbb{R}^n_{\min}$ can be formulated as finding $x \in \mathbb{R}^n_{\min}$ non zero, and $\lambda \in \mathbb{R}_{\min}$ such that :

$$\lambda x = f(x) \ .$$

Since f is 1-homogeneous, supposing without loss of generality that if x exists $x_1 \neq \epsilon$, the eigenvalue problem becomes :

$$\begin{cases} \lambda & = f_1(x/x_1) \ , \\ x_2/x_1 & = (f_2/f_1)(x) \ , \\ \cdots & = \cdots \\ x_n/x_1 & = (f_n/f_1)(x) \ , \end{cases}$$

Denoting $y = (x_2/x_1, \cdots, x_n/x_1)$ the problem is reduced to the computation of the fixed point problem $y = g(y)$ (with $g_{i-1}(y) = (f_i/f_1)(0, y)$) to compute a normalized eigenvector from which the eigenvalue is deduced by : $\lambda = f_1(0, y)$. But now g is a general minplus function.

The fixed point problem has not always a solution. There are cases where we are able to solve the problem – f is affine in standard algebra, – f is minplus linear, – f is positive power function. In the first case there is a unique eigenvalue as soon as $\dim(\ker(f - I)) = 1$.

In the two last cases, the problem can be reduced to the minimization of the average cost by time unit using dynamic programming methods. The corresponding fixed points are unique and stable.

Moreover, since $\max(x, y) = xy/(x \oplus y)$ games problem are also 1-homogeneous minplus systems and the solution of the corresponding eigenvalue problem is known.

In the general case we may have unstable fixed points that, nevertheless, we can compute by Newton method (which is exactly the policy iteration) but which don't give the information about the asymptotic behavior of the system anymore. In this case the asymptotic is obtained by an averaging based on invariant measure which may be difficult to compute. Let us give an example of chaotic system which has a 1-homogeneous minplus dynamics.

5. A Chaotic system example

Let us consider the 1-homogeneous minplus dynamic system

$$\begin{cases} x_1^{k+1} = (x_1^k)^2/x_2^k \oplus 2(x_2^k)^3/(x_1^k)^2 \ , \\ x_2^{k+1} = x_2^k \ . \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} \lambda x_1 = x_1^2/x_2 \oplus 2x_2^3/x_1^2, \\ \lambda x_2 = x_2 \ . \end{cases}$$

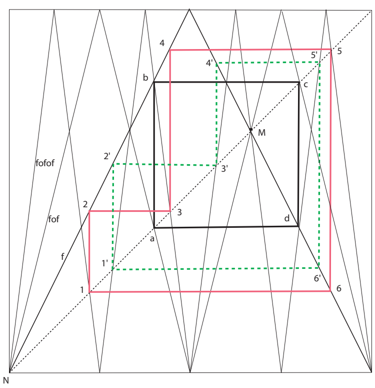

FIGURE 8. Cycles of tent transformation.

The solutions are $\lambda = 0$ and $y = x_1/x_2$ satisfying the equation

$$y = y^2 \oplus 2/y^2,$$

which has for solutions $y = 0$ and $y = 2/3$. These two solutions are unstable fixed points of the transformation $f(y) = y^2 \oplus 2/y^2$. But the system $y_{n+1} = f(y_n)$ is a chaotic system since f is the tent transform (see [**2**] for example for a clear discussion of this dynamics). In Figure 8 we show the graph of f, $f \circ f$, $f \circ f \circ f$, their fixed points and the corresponding periodic trajectories.

In Figure 9 we show a trajectory for an initial condition chosen randomly with the uniform law on the set $\{(i-1)/10^5, i = 1, \cdots, 10^5\}$. The diagonal line in the picture is a decreasing sort applied to the trajectory. It shows that the invariant empirical density is uniform. We can prove that the tent

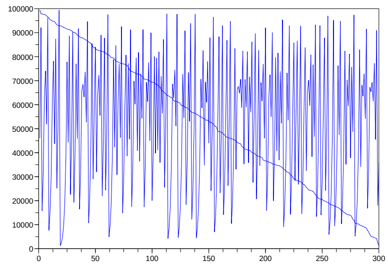

FIGURE 9. A tent iteration trajectory.

iteration has a unique invariant measure absolutely continuous with respect to the Lebesgue measure : the uniform law on $[0, 1]$.

More generally a chaotic 1-homogeneous minplus system will grow linearly with a value λ given by :

$$\lambda = \int f_1(y) d\mu(y) \ ,$$

where μ is the invariant probability measure of y depending on the initial value y^0. For example, according to the initial value y^0, the tent iterations y^k stay in circuits or follow trajectories without circuit (possibly dense in $[0, 1]$).

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Other informations and articles about this max-plus algebra are available from the web page : http://maxplus.org.