On Analytical Derivation of Traffic Phase Diagrams

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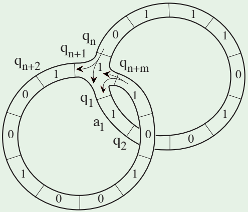
Outline

Aim of the talk

Traffic on two circular roads with one junction:



Question:

Can we derive analytically the average flow as a function of the car density (called the fundamental traffic diagram) ?

Main result

Answer:

We can give a very good analytic approximation by computing explicitly the eigenvalue of the dynamics which is minplus homogenous of degree 1 but not monotone.

Petri net Modeling

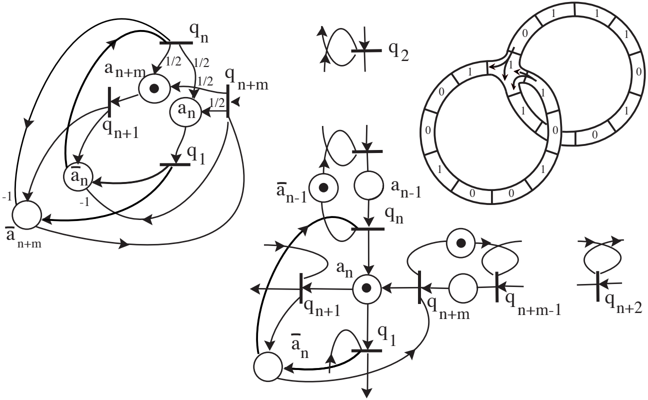


Figure: A junction with two circular roads cut in sections (top-right), its Petri net simplified modeling (middle) and the precise modeling of the junction (top left).

Dynamics of Petri net with conflict

The general Petri net equation:

$$\min_{p\in q^{in}}\left\{a_p+\sum_{q\in p^{in}} m_{pq}q^{k-1}-\sum_{q\in p^{out}} q^k\right\}=0,\ \forall q\in\mathcal{Q},\ \forall k,$$

does not define completely the dynamics.
The weights are number (m_{pq}) associated with the arcs living the transitions. In Petri net literature the weights are positive.

Conflict resolution using negative weights 1

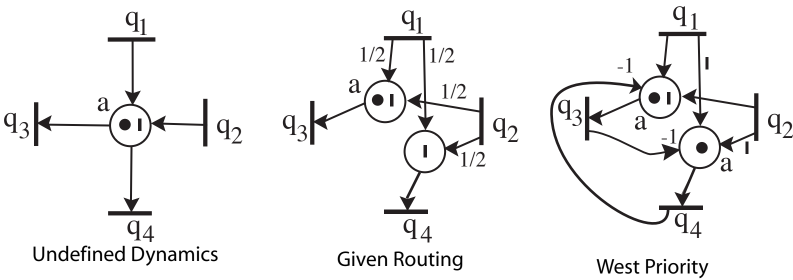


Figure: The Petri net with conflict, given in the left figure, is made clear by: – choosing a routing policy: 1/2 towards q_3, 1/2 towards q_4 in the central figure, – giving top priority to q_3 against q_4 in the right figure.

Conflict resolution using negative weights 2

The constraint expressed by the Peti net with conflict is :

$$q_4^k q_3^k = a q_1^{k-1} q_2^{k-1} .$$

clearly q_3 and q_4 are not defined uniquely. To define them uniquely we can clarify the dynamics by :

- Specify the routing policy

$$q_4^k = \sqrt{q_1^{k-1} q_2^{k-1}}, \quad q_3^k = a q_4^k .$$

- Choose a priority rule (top priority to q_3 against q_4)

$$q_3^k = a q_1^{k-1} q_2^{k-1} / q_4^{k-1}, \quad q_4^k = a q_1^{k-1} q_2^{k-1} / q_3^k .$$

Traffic System Dynamics

We precise the dynamics of the traffic system by giving the turning possibilities (1/2 go straight on 1/2 turn) for leaving the junction and the right priority to enter in the junction.

$$\begin{cases} q_i^{k+1} = a_{i-1} q_{i-1}^k \oplus \bar{a}_i q_{i+1}^k, \; i \neq 1, n, n+1, n+m, \\ q_n^{k+1} = \bar{a}_n \frac{q_1^k q_{n+1}^k}{q_{n+m}^k} \oplus a_{n-1} q_{n-1}^k \;, \\ q_{n+m}^{k+1} = \bar{a}_{n+m} \frac{q_1^k q_{n+1}^k}{q_n^{k+1}} \oplus a_{n+m-1} q_{n+m-1}^k \;, \\ q_1^{k+1} = a_n \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_1 q_2^k \;, \\ q_{n+1}^{k+1} = a_{n+m} \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_{n+1} q_{n+2}^k \;. \end{cases}$$

This dynamics is homogeneous of degree 1 but not monotone.

The Fundamental Traffic Diagram

We want the average flow as the function of the vehicle density :
Assuming that $\lim_N q_i^N/N$ does not depend of i we call χ that limit.
Denoting the car density $d = \sum_i a_i/(n+m-1)$, we want χ as a function of d.

Fundamental Traffic diagram

$$\chi(d)?$$

Definitions

Minplus homogeneous dynamical systems:

$$x^{k+1} = f(x^k), \text{ with } f : \mathbb{R}^n_{\min} \mapsto \mathbb{R}^n_{\min} : \; f(\lambda \otimes x) = \lambda \otimes f(x) \; .$$

Growth rate $\chi \in \mathbb{R}_{\min}$:

$$\chi = \lim_k x_i^k / k, \quad \forall i = 1, \cdots, n \; .$$

Eigenvalues $\lambda \in \mathbb{R}_{\min}$:

$$\exists x \neq \varepsilon : f(x) = \lambda \otimes x.$$

Problems and Applications

Questions:

$$?\exists \chi, \quad , ?\exists \lambda, \quad ?\chi = \lambda \,.$$

TRUE when f is monotone and $\mathcal{G}(f)$ strongly connected (Gaubert-Gunawerdena). What happens for general homogeneous system ?

Canonical form of Homogeneous Systems

The dynamics $x^{k+1} = f(x^k)$ is equivalent to

$$\begin{cases} x_1^{k+1}/x_1^k = f_1(x^k)/x_1^k, \\ x_i^{k+1}/x_1^{k+1} = f_i(x^k)/f_1(x^k), \quad i = 2, \cdots, n, \end{cases}$$

using the homogeneity it can be written :

Dynamics Canonical Form

$$\begin{cases} \Delta^k = h(y^k), \\ y^{k+1} = g(y^k), \end{cases}$$

with $\Delta^k \triangleq x_1^{k+1}/x_1^k$, $y_{i-1}^k = x_i^k/x_1^k$ and $g_{i-1} = f_i/f_1$ for $i = 2, \cdots, n$.

Growth rate

As soon as the y^k belong to a bounded closed (compact) set for all k, the set of measures:

$$\left\{ P^N_{y^0} = \frac{1}{N}\left(\delta_{y^0} + \delta_{g(y^0)} + \cdots + \delta_{g^{N-1}(y^0)}\right),\ N \in \mathbb{N} \right\} ,$$

is tight. Therefore we can extract convergent subsequences which converge towards invariant measures Q_{y^0}.

Applying the ergodic theorem to the sequence $(y^k)_{k\in\mathbb{N}}$:

Growth Rate Existence

$$\chi = \lim_N \frac{1}{N}(x_1^N - x_1^0) = \lim_N \frac{1}{N}\left(\sum_{k=0}^{N-1} h(y^k)\right) = \int h(y)dQ_{y^0}(y),\ Q_{y^0} \text{ a.e.}$$

Remarks on Growth rate Existence

It would be very useful to prove that the limit exists for sequence starting from y^0.

1. A priori homogeneous systems have not the uniform continuity property necessary to prove the convergence of the Cesaro means for sequence starting from y^0.
2. In the case where the compact set is finite, we can apply the ergodicity results on Markov chains with a finite state number to show the convergence of $P_{y^0}^N$ towards Q_{y^0} which proves the convergence of the Birkhoff average for the sequence starting from y^0.

Non Everywhere Convergence of Birkhoff Averages

$f : x \in \mathbb{T}^1 \to 2x \in \mathbb{T}^1$ with: $x^0 = 0.1\overset{2}{0}01\overset{4}{1}1100000\overset{8}{0}000\cdots$

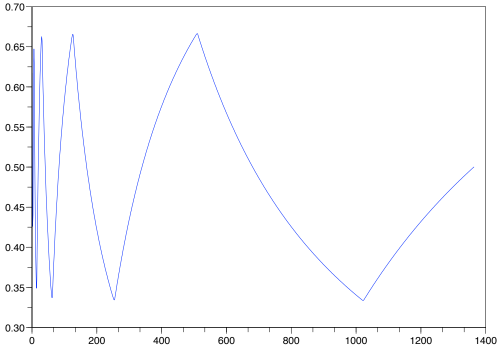


Figure: Plot of $S(n)$ with: $S(n) \triangleq \frac{1}{n}\sum_{k=0}^{n-1} x^k$.

Eigenvalue of Homogeneous Systems

The eigenvalue problem a function $f : \mathbb{R}^n_{\min} \mapsto \mathbb{R}^n_{\min}$ can be formulated as finding $x \in \mathbb{R}^n_{\min}$ non zero, and $\lambda \in \mathbb{R}_{\min}$ such that:

$$\lambda \otimes x = f(x) \ .$$

Since f is homogeneous, we can suppose without loss of generality that if x exists then $x_1 \neq \varepsilon$ and we have the:

Eigenvalue Canonical Form:

$$\begin{cases} \lambda & = h(y) \ , \\ y & = g(y) \ , \end{cases}$$

with $y_{i-1} = x_i / x_1$, $h(y) = f_1(x)/x_1$ and $g_{i-1} = f_i / f_1$ for $i = 2, \cdots, n$.

Eigenvalue Existence

Eigenvector Existence

The existence of eigenvalue is reduced to the existence of the fixed point of g which gives an eigenvector.

Standard Examples

1. f is a finite Markov chain transition operator.
2. f is affine in standard algebra with $dim(ker(f' - I_d)) = 1$.
3. f is minplus linear.
4. f is a dynamic programming function associated to a stochastic control problem.
5. f is a dynamic programming function associated to a stochastic game problem.

Affine Example with $dim(ker(f' - I_d)) = 1$.

With standard notations we have to solve:

$$\lambda + x = Mx + b, \; M\mathbf{1} = \mathbf{1}, \quad \text{Eigenvalue 1 simple} .$$

Using the variable change $z = Px$ with:

$$z = \begin{bmatrix} x_1 \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ -1 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & \cdot & \cdot & 1 \end{bmatrix} x .$$

The system $\lambda P\mathbf{1} + z = PMP^{-1}z + Pb$ has a block triangular form $PMP^{-1} = \begin{bmatrix} 1 & c \\ 0 & N \end{bmatrix}$ (thanks to the homogeneity $M\mathbf{1} = \mathbf{1}$), N has not the eigenvalue 1 (since 1 is a simple eigenvalue of PMP^{-1}) and therefore g has a unique fixed point.

Tent Example

Let us consider the homogeneous system:

$$\begin{cases} x_1^{k+1} = x_2^k \ , \\ x_2^{k+1} = (x_2^k)^3/(x_1^k)^2 \oplus 2(x_1^k)^2/x_2^k \ . \end{cases}$$

We have $h(y) = y$ and $g(y) = y^2 \oplus 2/y^2$ (g is the tent transformation which is chaotic).

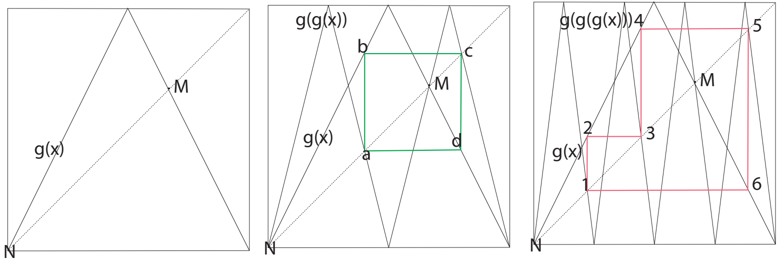


Figure: Tent transformation and its iterates.

$\chi \neq \lambda$

The eigenvalues are $\lambda = y$ solution of $y = y^2 \oplus 2/y^2$ that is:

$$\lambda \in \left\{0, \frac{2}{3}\right\}.$$

1. Starting from $y^0 = \frac{2}{5}$, the trajectory is periodic of period 2. The invariant measure is $Q_{y^0} = \frac{1}{2}(\delta_{\frac{2}{5}} + \delta_{\frac{6}{5}})$, therefore:

$$\chi = \frac{4}{5}, \quad Q_{y^0} \text{ a.e.}$$

2. The tent transformation admits the uniform law as invariant measure, therefore:

$$\chi = \int_0^1 y dy = \frac{1}{2}, \quad \text{a.e. for the Lebesgue measure.}$$

Increasing Property of Traffic Trajectories

Theorem

The trajectories of the states $(q_i^k)_{k\in\mathbb{N}}$, starting from 0, are nondecreasing for all i.

Proof by induction. For q_n:

If $q_n^{k+1} = a_{n-1} q_{n-1}^k$

$$\Rightarrow q_n^{k+1} \geq a_{n-1} q_{n-1}^{k-1} \geq f_n(q^{k-1}) = q_n^k \; .$$

If $q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$,

$$\Rightarrow q_n^{k+1} \geq \bar{a}_n q_n^k q_1^k q_{n+1}^k / \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1}$$

since $q_{n+m}^k \leq \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1} / q_n^k$

$$\Rightarrow q_n^{k+1} \geq q_n^k$$

Distances between states stay bounded

Theorem

The distances between any pair of states stay bounded:

$$\exists c_1 : \sup_k |q_i^k - q_j^k| \leq c_1, \forall i, j.$$

Moreover:

$$\forall T, \exists c_2 : \sup_k |q_i^{k+T} - q_i^k| \leq c_2 T, \forall i.$$

Existence of the Growth Rate

Theorem

There exists an initial distribution on $(q_j^0/q_1^0)_{j=2,n+m}$, the Kryloff Bogoljuboff invariant measure, such that the average flow

$$\chi = \lim_k q_i^k/k, \ \ \forall i \ ,$$

exists almost everywhere.

Eigenvalue Formula

The eigenvalue problem can be solved explicitly.

Theorem

The nonnegative eigenvalues λ *are given by:*

d	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\beta \wedge \gamma < d < \beta \vee \gamma$	$\gamma \leq d \leq 1$
λ	$(1-\rho)d$	$1/4$	$\frac{r-(1-\rho)d}{2r-1+2\rho}$	0

with $N = n + m$, $\rho = 1/N$, $r = m/N$, $\alpha = 1/4(1-\rho)$, $\beta = (r + 1/2 - \rho)/2(1-\rho)$ *and* $\gamma = r/(1-\rho)$ $\square$

$N >> 1, \; r > 1/2$

$$\lambda \simeq \max\left\{0, \; \min\left\{d, \frac{1}{4}, \frac{r-d}{2r-1}\right\}\right\}.$$

Difference Between Eigenvalue and Growth Rate

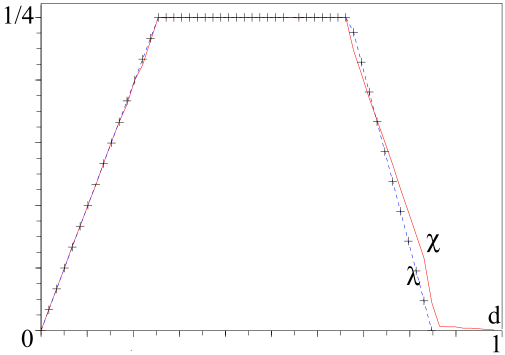


Figure: The traffic fundamental diagram $\chi(d)$ when $r = 5/6$ (continuous line) obtained by simulation and its comparison with the eigenvalue $\lambda(d)$.

Phases

1. *Free moving:* When the density is small, $0 \leq d \leq \alpha$ with $\alpha = \frac{1}{4(1-\rho)}$, after a finite time, all the cars move freely.
2. *Saturation:* When $\alpha \leq d \leq \beta$ with $\beta = \frac{1}{2}\frac{r+1/2-\rho}{1-\rho}$ the junction is used at its maximal capacity without being bothered by downstream cars.
3. *Recession:* When $\beta < d < \gamma$ with $\gamma = \frac{r}{1-\rho}$ the crossing is fully occupied but cars sometimes cannot leave it because the roads where they want to go are crowded. When $\gamma < \beta$, on the interval $[\gamma, \beta]$ three eigenvalues exist. In this case the system is in fact blocked.
4. *Blocking:* When $\gamma \leq d \leq 1$, the road without priority is full of cars, no car can leave it and one car wants to enter.

The Eigenvalue Problem

We have to solve the system:

$$\begin{cases} \lambda q_i = a_{i-1} q_{i-1} \oplus \bar{a}_i q_{i+1}, \ i \neq 1, n, n+1, n+m, \\ \lambda q_n = \bar{a}_n q_1 q_{n+1}/q_{n+m} \oplus a_{n-1} q_{n-1} \ , \\ \lambda q_{n+m} = \bar{a}_{n+m} q_1 q_{n+1}/(\lambda q_n) \oplus a_{n+m-1} q_{n+m-1} \ , \\ \lambda q_1 = a_n \sqrt{q_n q_{n+m}} \oplus \bar{a}_1 q_2 \ , \\ \lambda q_{n+1} = a_{n+m} \sqrt{q_n q_{n+m}} \oplus \bar{a}_{n+1} q_{n+2} \ , \end{cases} \tag{1}$$

with: $0 \leq a_i \leq 1$ for $i = 1, \cdots, n+m$, $\bar{a}_i = 1 - a_i$ for $i \neq n, n+m$, $a_n + a_{n+m} \leq 1$ and $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$.

Reduction

This result is obtained by reducing the system of size N to a system of size 4. For that we eliminate $q_i, i \neq 1, n, n+1, n+m$. For that, using $\lambda \leq 1/4$ (see the proof later) we can verify that for $i = 2, n-1$:

$$q_i = q_1 \left[\bigotimes_{j=1}^{i-1} (a_j/\lambda) \right] \oplus \left[\bigotimes_{j=i}^{n-1} (\bar{a}_j/\lambda) \right] q_n \ ,$$

for $i = n+1, n+m-1$:

$$q_i = q_{n+1} \left[\bigotimes_{j=n+1}^{i-1} (a_j/\lambda) \right] \oplus \left[\bigotimes_{j=i}^{n+m-1} (\bar{a}_j/\lambda) \right] q_{n+m} \ .$$

The Reduced System

Using simpler variable and constant names : $U = q_n$, $V = q_{n+m}$, $X = q_1$, $Y = q_{n+1}$, $n' = n - 1$, $m' = m - 1$, g and h being the number of cars in the two roads, the reduced system can be written :

$$\begin{cases} U = \bar{k}XY/\lambda V \oplus gX/\lambda^{n'} \ , \\ V = \bar{k}XY/\lambda^2 U \oplus hY/\lambda^{m'} \ , \\ X = k\sqrt{UV}/\lambda \oplus \bar{g}U/\lambda^{n'} \ , \\ Y = l\sqrt{UV}/\lambda \oplus \bar{h}V/\lambda^{m'} \ , \end{cases} \tag{2}$$

with $\bar{k} = 1 - k - l \geq 0$, $\bar{g} = n' - g \geq 0$ and $\bar{h} = m' - h \geq 0$ are the free places in the crossing and in the two roads respectively.

Resolution of the reduced system

As soon as we know on what "monomial" the min is reached, the system is reduced to a standard affine system of 4 equations and 5 unknowns. The physical interpretation of the four phases helps to find this monomial.
When we have guessed where the min is reached in each equation, we have only to verify that the other monomial is greater.

The free phase case verification

Let us verify the case $0 \leq d \leq \alpha$. $U = g/\lambda^{n'}$, $V = \lambda^{n'+2}/k^2 g = hl/k\lambda^{m'}$, $X = e \triangleq 0$, $Y = l/k$ is a solution solution of the standard linear system :

$$U = gX/\lambda^{n'},\ V = hY/\lambda^{m'},\ X = k\sqrt{UV}/\lambda\ ,\ Y = l\sqrt{UV}/\lambda\ , \tag{3}$$

which is itself solution of System (2) since :

- $\bar{k}XY/\lambda VU = \bar{k}kl/\lambda^3 = 1/\lambda^3 \geq 0$ (since $\lambda \leq 1/4$ indeed using (2) we have $VXY \leq \bar{k}klVXY/\lambda^4$) ,
- $\bar{k}XY/\lambda^2 VU = 1/\lambda^4 \geq 0$,
- $\bar{g}U/X\lambda^{n'} = (1/\lambda^2)^{n'} \geq 0$,
- $\bar{h}V/Y\lambda^{m'} = \bar{h}h/\lambda^{m'} = (1/\lambda^2)^{m'} \geq 0$.

Moreover multiplying the 4 equalities of (3) we obtain $\lambda^{n'+m'+2} = ghlk$ which gives the value of λ.

Extension to Regular Towns

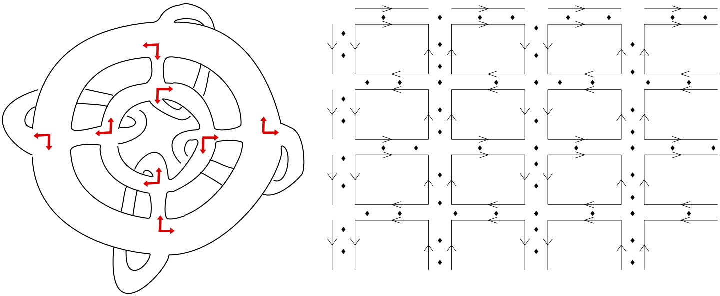


Figure: Roads on a torus of 4×2 streets with its authorized turn at junctions (left) and the asymptotic car repartition in the streets on a torus of 4×4 streets obtained by simulation.

Light Controlled Fundamental Diagram

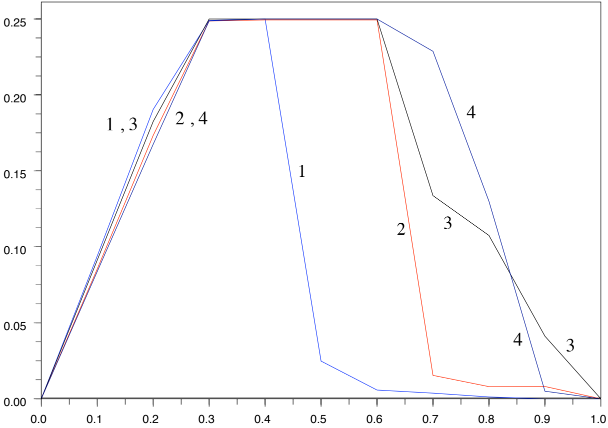


Figure: Light policies comparison for a regular town on a torus. (1) right priority, (2) open loop, (3) local feedback, (4) global feedback obtained by LQ methods.

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