MAX-PLUS CONVEX SETS AND FUNCTIONS

GUY COHEN, STÉPHANE GAUBERT, JEAN-PIERRE QUADRAT, AND IVAN SINGER

Abstract. We consider convex sets and functions over idempotent semifields, like the max-plus semifield. We show that if $\mathcal{K}$ is a conditionally complete idempotent semifield, with completion $\bar{\mathcal{K}}$, a convex function $\mathcal{K}^n \to \bar{\mathcal{K}}$ which is lower semi-continuous in the order topology is the upper hull of supporting functions defined as residuated differences of affine functions. This result is proved using a separation theorem for closed convex subsets of $\mathcal{K}^n$, which extends earlier results of Zimmermann, Samborski, and Shpiz.

Contents

1. Introduction

In this paper, we consider convex subsets of semimodules over semirings with an idempotent addition, like the max-plus semifield $\mathbb{R}_{\max}$, which is the set $\mathbb{R} \cup \{-\infty\}$, with $(a, b) \mapsto \max(a, b)$ as addition, and $(a, b) \mapsto a + b$ as multiplication. Convex subsets $C \subset \mathbb{R}^n_{\max}$, or *max-plus convex sets*, satisfy

$$(x, y \in C,\ \alpha, \beta \in \mathbb{R}_{\max},\ \max(\alpha, \beta) = 0) \implies \max(\alpha + x, \beta + y) \in C\ ,$$

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where the operation "max" should be understood componentwise, and where $\alpha+x = (\alpha + x_1, \dots, \alpha + x_n)$ for $x = (x_1, \dots, x_n)$. We say that a function $f : \mathbb{R}^n_{\max} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is *max-plus convex* if its epigraph is max-plus convex. An example of max-plus convex function is depicted in Figure 1 (further explanations will be given in §4).

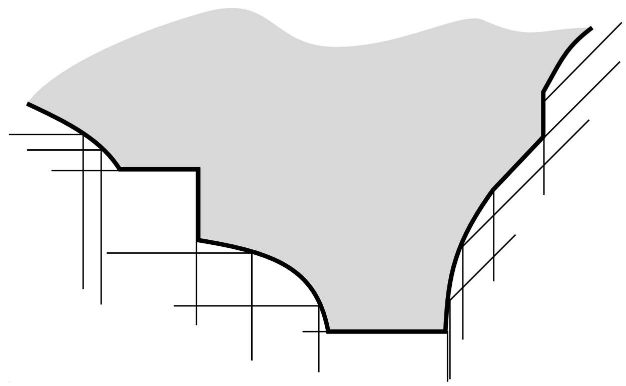

FIGURE 1. A convex function over $\mathbb{R}_{\max}$ and its supporting half spaces

Motivations to study semimodules and convex sets over idempotent semirings arise from several fields. First, semimodules over idempotent semirings, which include as special cases sup-semilattices with a bottom element (wich are semimodules over the Boolean semiring), are natural objects in lattice theory. A second motivation arises from dynamic programming and discrete optimization. Early results in this direction are due to Cuninghame-Green (see [CG79]), Vorobyev [Vor67, Vor70], Romanovski [Rom67], K. Zimmermann [Zim76]. The role of max-plus algebra in Hamilton-Jacobi equations and quasi-classical asymptotics, discovered by Maslov [Mas73, Ch. VII] led to the development of an "idempotent analysis", by Kolokoltsov, Litvinov, Maslov, Samborski, Shpiz, and others (see [MS92, KM97, LMS01] and the references therein). A third motivation arises from the algebraic approach of discrete event systems [BCOQ92]: control problems for discrete event systems are naturally expressed in terms of invariant spaces [CGQ99].

Another motivation, directly related to the present work, comes from abstract convex analysis [Sin84, Sin97, Rub00]: a basic result of convex analysis states that convex lower semi-continuous functions are upper hulls of affine maps, which means precisely that the set of convex functions is the max-plus (complete) semimodule generated by linear maps. In the theory of generalized conjugacies, linear maps are replaced by a general family of maps, and the set of convex functions is replaced by a general semimodule. More precisely, given an abstract class of convex sets and functions, a basic issue is to find a class of elementary functions with which convex sets and functions can be represented. This can be formalized in terms of *U-convexity* [DK78, Sin97]. If X is a set and $U \subset \bar{\mathbb{R}}^X$, a function $f : X \to \bar{\mathbb{R}}$ is called *U-convex* if there exists a subset U' of U such that

$$f(x) = \sup_{u \in U'} u(x) \ , \qquad \forall x \in X \ . \tag{1}$$

A subset C of X is said to be U-*convex* [Fan63, Sin97] if for each $y \in X \setminus C$ we can find a map $u \in U$ such that

$$u(y) > \sup_{x\in C} u(x) \quad . \tag{2}$$

In this paper, we address the problem of finding the set U adapted to max-plus convex sets and functions. The analogy with classical algebra suggests to introduce *max-plus linear* functions:

$$\langle a, x\rangle = \max_{1\le i\le n}(a_i + x_i) \quad , \tag{3}$$

with $a = (a_i) \in \mathbb{R}^n_{\max}$, and *max-plus affine* functions, which are of the form

$$u(x) = \max(\langle a, x\rangle, b) \quad , \tag{4}$$

where $b \in \mathbb{R}_{\max}$. In the max-plus case, we cannot take for U the set of affine or linear functions, because any sup of max-plus affine (resp. linear) functions remains max-plus affine (resp. linear). This is illustrated in the last (bottom right) picture in Figure 2, which shows the graph of a generic affine function in dimension 1 (the graph is the black broken line, see Table 1 in §4 for details). It is geometrically obvious that we cannot obtain the convex function of Figure 1 as the sup of affine functions. (Linear functions, however, lead to an interesting theory if we consider max-plus *concave* functions instead of max-plus convex functions, see Rubinov and Singer [RS00], and downward sets instead of max-plus convex sets, see Martínez-Legaz, Rubinov, and Singer [MLRS02].)

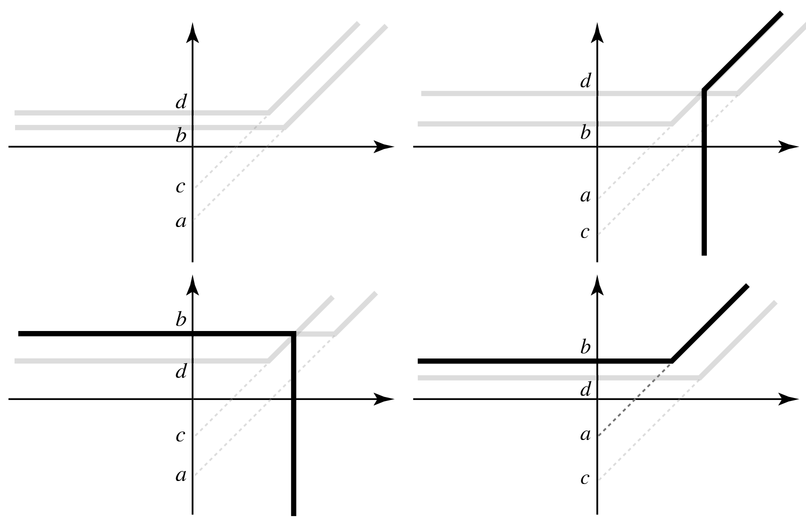

FIGURE 2. The four generic differences of affine functions plots

We show here that for max-plus convex functions, and more generally for convex functions over conditionally complete idempotent semifields, the appropriate U consists of *residuated differences* of affine functions, which are of the form $u \ominus u'$, where u, u' are affine functions, and $\ominus$ denotes the residuated law of the semiring addition, defined in (10) below. Theorem 4.8 shows that lower semi-continuous convex functions are precisely upper hulls of residuated differences of affine functions, and Corollary 4.7 shows that the corresponding U-convex sets are precisely the closed convex sets.

As an illustration, in the case of the max-plus semiring, in dimension 1, there are 4 kinds of residuated differences of affine functions, as shown in Figure 2, and Table 1: one of these types consists of affine functions (bottom right, already discussed), another of these types consists only of the identically $-\infty$ function (top left, not visible), whereas the top right and bottom left plots yield new shapes, which yield "supporting half-spaces" for the convex function of Figure 1.

The main device in the proof of these results is a separation theorem for closed convex sets (Theorem 3.14). We consider a convex subset C of $\mathcal{K}^n$, where $\mathcal{K}$ is an idempotent semifield that is conditionally complete for its natural order. Then, we show that if C is stable under taking sups of (bounded) directed subsets and infs of (bounded) filtered subsets, or equivalently, if C is closed in Birkhoff's order topology, and if $y \in \mathcal{K}^n \setminus C$, there exists an affine hyperplane

$$H = \{x \in \mathcal{K}^n \mid u(x) = u'(x)\} \ ,$$

with u, u' as in (4), containing C and not y. When $\mathcal{K} = \mathbb{R}_{\max}$, Birkhoff's order topology coincides with the usual one, and we get a separation theorem for convex subsets of $\mathbb{R}^n_{\max}$ which are closed in the usual sense. The key discrepancy, by comparison with usual convex sets, is that a two sided equation $u(x) = u'(x)$ is needed. Theorem 3.14 extends or refines earlier results by Zimmermann [Zim77], Samborski and Shpiz [SS92], and by the three first authors [CGQ02]. Some metric assumptions on the semifield, which were used in [Zim77], are eliminated, and the proof of Theorem 3.14 is in our view simpler (with a direct geometric interpretation in terms of projections). The method of [SS92] only applies to the case where the vector y does not have entries equal to the bottom element. This restriction is removed in Theorem 3.14 (see Example 3.18 below for details). By comparison with [CGQ02], the difference is that we work here in conditionally complete semifields (without a top element), whereas the result of [CGQ02] applies to the case of complete semirings (which necessarily have a top element). When the top element is a coefficient of an affine equation defining an hyperplane, the hyperplane need not be closed in the order topology, and a key part of the proof of Theorem 3.14 is precisely to eliminate the top element from the equation defining the hyperplanes.

We finally point out additional references in which semimodules over idempotent semirings or related structures appear: [Kor65, Zim81, CKR84, Wag91, Gol92, CGQ96, CGQ97, LS02, GM02].

2. Preliminaries

2.1. Ordered sets, residuation, idempotent semirings and semimodules.

In this section, we recall some basic notions about partially ordered sets, residuation, idempotent semirings and semimodules. See [Bir67, DJLC53, BJ72, CGQ02] for more details. By *ordered set*, we will mean throughout the paper a set equipped with a *partial* order. We say that an ordered set $(S, \leq)$ is *complete* if any subset $X \subset S$ has a least upper bound (denoted by $\bigvee X$). In particular, S has both a minimal (bottom) element $\bot S = \bigvee \emptyset$, and a maximal (top) element $\top S = \bigvee S$. Since the greatest lower bound of a subset $X \subset S$ can be defined by $\bigwedge X = \bigvee \{y \in S \mid y \leq x, \ \forall x \in X\}$, S is a complete lattice. We shall also consider the case where

S is only *conditionally complete*, which means that any subset of S bounded from above has a least upper bound and that any subset of S bounded from below has a greatest lower bound.

If $(S,\leq)$ and $(T,\leq)$ are ordered sets, we say that a map $f:S\to T$ is *residuated* if there exists a map $f^\sharp:T\to S$ such that

$$f(s)\leq t \iff s\leq f^\sharp(t) \enspace , \tag{5}$$

which means that for all $t\in T$, the set $\{s\in S\mid f(s)\leq t\}$ has a maximal element, $f^\sharp(t)$. If $(X,\leq)$ is an ordered set, we denote by $(X^{\mathrm{op}},\overset{\mathrm{op}}{\leq})$ the *opposite* ordered set, for which $x\overset{\mathrm{op}}{\leq}y \iff x\geq y$. Due to the symmetry of the defining property (5), it is clear that if $f:S\to T$ is residuated, then $f^\sharp:T^{\mathrm{op}}\to S^{\mathrm{op}}$ is also residuated. When S,T are complete ordered sets, there is a simple characterization of residuated maps. We say that a map $f:S\to T$ *preserves arbitrary sups* if for all $U\subset S$, $f(\vee U)=\vee f(U)$, where $f(U)=\{f(x)\mid x\in U\}$. In particular, when $U=\emptyset$, we get $f(\bot S)=\bot T$. One easily checks that if $(S,\leq)$ and $(T,\leq)$ are complete ordered sets, then, a map $f:S\to T$ is residuated if, and only if, it preserves arbitrary sups (see [BJ72, Th. 5.2], or [BCOQ92, Th. 4.50]). In particular, a residuated map f is *isotone*, $x\leq y \implies f(x)\leq f(y)$, which, together with (5), yields $f\circ f^\sharp\leq I$ and $f^\sharp\circ f\geq I$. This also implies that:

$$f\circ f^\sharp\circ f=f, \qquad f^\sharp\circ f\circ f^\sharp=f^\sharp \enspace . \tag{6}$$

We now apply these notions to idempotent semirings and semimodules. Recall that a *semiring* is a set $\mathcal{S}$ equipped with an addition $\oplus$ and a multiplication $\otimes$, such that $\mathcal{S}$ is a commutative monoid for addition, $\mathcal{S}$ is a monoid for multiplication, multiplication left and right distributes over addition, and the zero element of addition, $\mathbb{0}$, is absorbing for multiplication. We denote by $\mathbb{1}$ the neutral element of multiplication (unit). We say that $\mathcal{S}$ is *idempotent* when $a\oplus a=a$. All the semirings considered in the sequel will be idempotent. We shall adopt the usual conventions, and write for instance ab instead of $a\otimes b$. An idempotent monoid $(S,\oplus,\mathbb{0})$ can be equipped with the *natural* order relation, $a\leq b\Leftrightarrow a\oplus b=b$, for which $a\oplus b=a\vee b$, and $\mathbb{0}=\bot\mathcal{S}$. We say that the semiring $\mathcal{S}$ is *complete* (resp. *conditionally complete*) if it is complete (resp. conditionally complete) as a naturally ordered set, and if for all $a\in\mathcal{S}$, the left and right multiplications operators, $\mathcal{S}\to\mathcal{S}$, $x\mapsto ax$, and $x\mapsto xa$, respectively, preserve arbitrary sups (resp. preserves sups of bounded from above sets). An *idempotent semifield* is an idempotent semiring whose nonzero elements are invertible. An idempotent semifield $\mathcal{S}$ cannot be complete, unless $\mathcal{S}$ is the two-element *Boolean semifield*, $\{\mathbb{0},\mathbb{1}\}$. However, a conditionally complete semifield $\mathcal{S}$ can be embedded in a complete semiring $\bar{\mathcal{S}}$, which is obtained by adjoining to $\mathcal{S}$ a top element, τ, and setting $a\oplus\tau=\tau$, $\mathbb{0}\tau=\tau\mathbb{0}=\mathbb{0}$, and $a\tau=\tau a=\tau$ for $a\neq\mathbb{0}$. Then, we say that $\bar{\mathcal{S}}$ is the *completed semiring* of $\mathcal{S}$ ($\bar{\mathcal{S}}$ was called the *top-completion* of $\mathcal{S}$ in [CGQ97], and the *minimal completion* of $\mathcal{S}$ in [AS03]). For instance, the *max-plus semifield* $\mathbb{R}_{\max}$, defined in the introduction, can be embedded in the *completed max-plus semiring* $\bar{\mathbb{R}}_{\max}$, whose set of elements is $\bar{\mathbb{R}}$.

A (right) $\mathcal{S}$-*semimodule* X is a commutative monoid $(X,\oplus,\mathbb{0})$, equipped with a map $X\times\mathcal{S}\to X$, $(x,\lambda)\to x\lambda$ (right action), that satisfies $x(\lambda\mu)=(x\lambda)\mu$, $(x\oplus y)\lambda=x\lambda\oplus y\lambda$, $x(\lambda\oplus\mu)=x\lambda\oplus x\mu$, $x\mathbb{0}=\mathbb{0}$, and $x\mathbb{1}=x$, for all $x,y\in X$, $\lambda,\mu\in\mathcal{S}$, see [CGQ02] for more details. Since $(\mathcal{S},\oplus)$ is idempotent, $(X,\oplus)$ is idempotent, so that $\oplus$ coincides with the $\vee$ law for the natural order of X. All the semimodules that we shall consider will be right semimodules over idempotent semirings. If $\mathcal{S}$ is a complete semiring, we shall say that a $\mathcal{S}$-semimodule X is *complete* if it is complete as a naturally ordered set, and if, for all $x \in X$ and $\lambda \in \mathcal{S}$, the left and right multiplications, $X \to X$, $x \mapsto x\lambda$, and $\mathcal{S} \to X$, $\mu \mapsto v\mu$, respectively, preserve arbitrary sups. We shall say that $V \subset X$ is a *complete subsemimodule* of X if V is a subsemimodule of X stable under arbitrary sups. A basic example of semimodule over an idempotent semiring $\mathcal{S}$ is the *free semimodule* $\mathcal{S}^n$, or more generally the semimodule $\mathcal{S}^I$ of functions from an arbitrary set I to $\mathcal{S}$, which is complete when $\mathcal{S}$ is complete. For $x \in \mathcal{S}^I$ and $i \in I$, we denote, as usual, by x_i the i-th entry of x.

In a complete semimodule X, we define, for all $x, y \in X$,

$$x \backslash y = \top\{\lambda \in \mathcal{S} \mid x\lambda \le y\} \ , \tag{7}$$

where we write $\top$ for the least upper bound to emphasize the fact that the set has a top element. In other words, $y \mapsto x \backslash y$, $\mathcal{S} \to \mathcal{S}$ is the residuated map of $\lambda \mapsto x\lambda$, $X \to X$. Specializing (5), we get

$$x\lambda \le y \iff \lambda \le x \backslash y \ . \tag{8}$$

For instance, when $\mathcal{S} = \bar{\mathbb{R}}_{\max}$, $(-\infty) \backslash (-\infty) = (+\infty) \backslash (+\infty) = +\infty$, and $\mu \backslash \nu = \nu - \mu$ if (μ, ν) takes other values ($\mathcal{S}$ being thought of as a semimodule over itself). More generally, if $\mathcal{S}$ is any complete semiring, the law "$\backslash$" of the semimodule $\mathcal{S}^n$ can be computed from the law "$\backslash$" of $\mathcal{S}$ by

$$x \backslash y = \bigwedge_{1 \le i \le n} x_i \backslash y_i \ . \tag{9}$$

Here, $\backslash$ has a higher priority than $\wedge$, so that the right hand side of (9) reads $\bigwedge_{1 \le i \le n} (x_i \backslash y_i)$. If the addition of $\mathcal{S}$ distributes over arbitrary infs (this is the case in particular if $\mathcal{S}$ is a semifield, or a completed semifield, see [Bir67, Ch. 12, Th. 25]), for all $\lambda \in \mathcal{S}$, the translation by λ, $\mu \mapsto \lambda \oplus \mu$, defines a residuated map $\mathcal{S}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}$, and we set:

$$\nu \ominus \lambda = \bot\{\mu \mid \lambda \oplus \mu \ge \nu\} \ , \tag{10}$$

where we write $\bot$ for the greatest lower bound to emphasize the fact that the set has a bottom element. When $\mathcal{S} = \bar{\mathbb{R}}_{\max}$, we have (see e.g. [BCOQ92, MLS91]):

$$\nu \ominus \mu = \begin{cases} \nu & \text{if } \nu > \mu, \\ -\infty & \text{otherwise.} \end{cases}$$

Dualizing the definition (5) of residuated maps, we get:

$$\lambda \oplus \mu \ge \nu \iff \lambda \ge \nu \ominus \mu \ . \tag{11}$$

2.2. **Separation theorem for complete convex sets.** We next recall the general separation theorem of [CGQ01, CGQ02]. By *complete semimodule*, we mean throughout the section a complete semimodule over a complete idempotent semiring $\mathcal{S}$. Let V denote a complete subsemimodule of a complete semimodule X. We call *canonical projector* onto V the map

$$P_V : X \to V, \quad P_V(x) = \top\{v \in V \mid v \le x\}$$

(the least upper bound of $\{v \in V \mid v \le x\}$ belongs to the set because V is complete). Thus, P_V is the residuated map of the canonical injection $i_V : V \to X$, P_V is surjective, and $P_V = P_V^2$. If $\{w_\ell\}_{\ell\in L} \subset X$ is an arbitrary family, we set

$$\bigoplus_{\ell\in L} w_\ell := \bigvee\{w_\ell \mid \ell \in L\} \enspace .$$

We say that W is a *generating family* of a complete subsemimodule V if any element $v \in V$ can be written as $v = \bigoplus_{w\in W} w\lambda_w$, for some $\lambda_w \in \mathcal{S}$. If V is a complete subsemimodule of X with generating family W, then

$$P_V(x) = \bigoplus_{w\in W} w(w \backslash x) \enspace , \tag{12}$$

see [CGQ02, Th. 5].

Theorem 2.1 (Universal Separation Theorem, [CGQ02, Th. 8])**.** *Let $V \subset X$ denote a complete subsemimodule, and let $y \in X \setminus V$. Then, the set*

$$H = \{x \in X \mid x \backslash P_V(y) = x \backslash y\} \tag{13}$$

contains V and not y.

Seeing $x \backslash y$ as a "scalar product", H can be seen as the "hyperplane" of vectors x "orthogonal" to $(y, P_V(y))$. As shown in [CGQ02], the "hyperplane" H is a complete subsemimodule of X, even if it is defined by a nonlinear equation. In order to give a linear defining equation for this hyperplane, we have to make additional assumptions on the semiring $\mathcal{S}$. In this paper, we shall assume that $\mathcal{S} = \bar{\mathcal{K}}$ is the completed semiring of a conditionally complete idempotent semifield $\mathcal{K}$. Consider the semimodule of functions $X = \bar{\mathcal{K}}^I$. When $x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \in \bar{\mathcal{K}}^I$, we define

$$\langle y, x\rangle = \bigoplus_{i\in I} y_i x_i \tag{14}$$

and

$$^{-}x = \top\{y \in \bar{\mathcal{K}}^n \mid \langle y, x\rangle \leq \mathbb{1}\}$$

that is,

$$(^{-}x)_i = x_i \backslash \mathbb{1} \enspace . \tag{15}$$

(For instance, when $\mathcal{K} = \mathbb{R}_{\max}$, $(^{-}x)_i = -x_i$.) We have $^{-}(x \backslash y) = \langle ^{-}y, x\rangle$, and $\lambda \mapsto {}^{-}\lambda$ is bijective $\bar{\mathcal{K}} \to \bar{\mathcal{K}}$, which allows us to write H linearly:

$$H = \{x \in \bar{\mathcal{K}}^n \mid \langle ^{-}P_V(y), x\rangle = \langle ^{-}y, x\rangle\} \tag{16}$$

(see [CGQ02] for generalizations to more general semirings, called *reflexive* semirings).

Theorem 2.1 yields a separation result for convex sets as a corollary. We recall that a subset C of a complete semimodule X over a complete semiring $\mathcal{S}$ is *convex* [Zim77, Zim79b] (resp. *complete convex* [CGQ02]) if for all finite (resp. arbitrary) families $\{x_\ell\}_{\ell\in L} \subset C$ and $\{\alpha_\ell\}_{\ell\in L} \subset \mathcal{K}$, such that $\bigoplus_{\ell\in L} \alpha_\ell = \mathbb{1}$, we have that $\bigoplus_{\ell\in L} x_\ell\alpha_\ell \in C$. For example, every subsemimodule of X is convex, and every complete subsemimodule of X is complete convex.

Corollary 2.2 (Separating a Point from a Complete Convex Set, [CGQ02, Cor. 15])**.** *If C is a complete convex subset of a complete semimodule X, and if $y \in X \setminus C$, then the set*

$$H = \{x \in X \mid x \backslash y \wedge \mathbb{1} = x \backslash Q_C(y) \wedge \nu_C(y)\} \tag{17}$$

with

$$\nu_C(y) = \bigvee_{v \in C} (v \backslash y \wedge \mathbb{1}) \qquad \text{and} \qquad Q_C(y) = \bigvee_{v \in C} v(v \backslash y \wedge \mathbb{1}) \; , \tag{18}$$

contains C *and not* y.

Recall our convention explained after Equation 9, that $\backslash$ has a higher priority than $\wedge$, so that for instance $v \backslash y \wedge \mathbb{1} = (v \backslash y) \wedge \mathbb{1}$. When $\mathcal{S} = \bar{\mathcal{K}}$ is a completed idempotent semifield, and $X = \bar{\mathcal{K}}^I$, H can be rewritten linearly:

$$H = \{x \in \bar{\mathcal{K}}^I \mid \langle {}^{-}y, x\rangle \oplus \mathbb{1} = \langle {}^{-}Q_C(y), x\rangle \oplus {}^{-}\nu_C(y)\} \; . \tag{19}$$

Remark 2.3. Since $Q_C(y) \leq y$, and $\nu_C(y) \leq \mathbb{1}$, we have ${}^{-}y \leq {}^{-}Q_C(y)$ and $\mathbb{1} = {}^{-}\mathbb{1} \leq {}^{-}\nu_C(y)$, and hence, by definition of the natural order $\leq$, we can write equivalently H as

$$H = \{x \in \bar{\mathcal{K}}^I \mid \langle {}^{-}y, x\rangle \oplus \mathbb{1} \geq \langle {}^{-}Q_C(y), x\rangle \oplus {}^{-}\nu_C(y)\} \; .$$

The same remark applies, mutatis mutandis, to (13), (16), and (17).

2.3. **Geometric interpretation.** We now complement the results of [CGQ02] by giving a geometric interpretation to the vector $Q_C(y)$ and scalar $\nu_C(y)$ which define the separating hyperplane H. If C is any subset of X, we call *shadow of* C, denoted by $\mathrm{Sh}(C)$, the set of linear combinations

$$\bigoplus_{\ell \in L} x_\ell \lambda_\ell, \quad \text{with} \quad x_\ell \in C, \; \lambda_\ell \in \mathcal{S}, \; \lambda_\ell \leq \mathbb{1} \; , \text{ and } L \text{ a possibly infinite set.}$$

We also denote by

$$\mathrm{Up}(C) = \{z \in C \mid \exists v \in C, \; z \geq v\}$$

the *upper set* generated by C. The term "shadow" can be interpreted geometrically: when for instance $C \subset \bar{\mathbb{R}}^2_{\max}$, $\mathrm{Sh}(C)$ is the shadow of C if the sun light comes from the top-right corner of the plane, see Figure 3 and Example 2.5 below.

Theorem 2.4 (Projection onto $\mathrm{Sh}(C)$ and C)**.** *If* C *is a complete convex subset of a complete semimodule* X*, then, for all* $y \in X$*,*

$$Q_C(y) = \top\{z \in \mathrm{Sh}(C) \mid z \leq y\} \; . \tag{20}$$

If $y \in \mathrm{Up}(C)$*,*

$$Q_C(y) = \top\{z \in C \mid z \leq y\}, \qquad \text{and} \qquad \nu_C(y) = \mathbb{1} \; . \tag{21}$$

If $\nu_C(y)$ *is invertible,* $Q_C(y)(\nu_C(y))^{-1}$ *belongs to* C*.*

Thus, Theorem 2.4 shows that Q_C is a projector which sends X to $\mathrm{Sh}(C)$, and $\mathrm{Up}(C)$ to C. Moreover, when $\nu_C(y)$ is invertible, $Q_C(y)(\nu_C(y))^{-1}$ can be considered as the projection of y onto C.

Proof. Since $v \backslash y \wedge \mathbb{1} \leq \mathbb{1}$,

$$Q_C(y) = \bigoplus_{v \in C} v(v \backslash y \wedge \mathbb{1}) \in \mathrm{Sh}(C) \; . \tag{22}$$

If $y \in \mathrm{Up}(C)$, we have $v \leq y$ for some $v \in C$, hence, $v \backslash y \geq \mathbb{1}$ (by (8)), which implies that $\nu_C(y) \geq v \backslash y \wedge \mathbb{1} = \mathbb{1}$. Since $\nu_C(y) \leq \mathbb{1}$ holds trivially, we have proved that $\nu_C(y) = \mathbb{1}$, so that

$$y \in \mathrm{Up}(C) \implies Q_C(y) \in C \text{ and } \nu_C(y) = \mathbb{1} \; . \tag{23}$$

Consider now any element $z \in \mathrm{Sh}(C)$, $z = \bigoplus_{\ell \in L} v_\ell \lambda_\ell$, with $v_\ell \in C$, $\lambda_\ell \in \mathcal{S}$, $\lambda_\ell \leq \mathbb{1}$, and assume that $z \leq y$. Then, $v_\ell \lambda_\ell \leq y$, so that $\lambda_\ell \leq v_\ell \backslash y$ (by (8)), and since

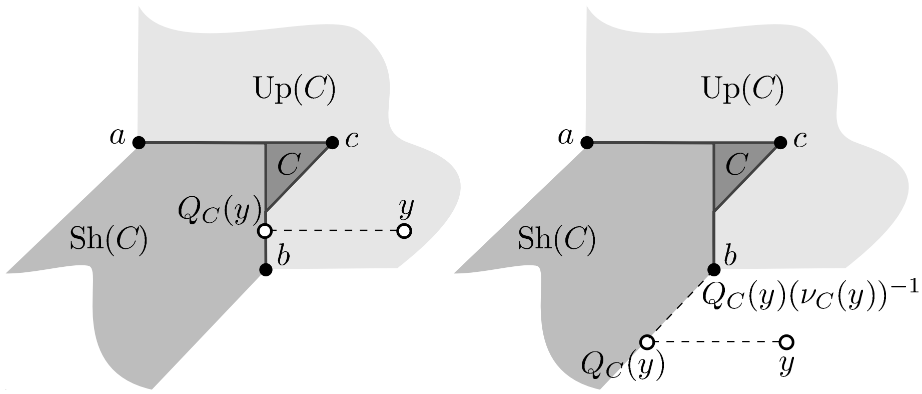

FIGURE 3. Projections

$\lambda_\ell \leq \mathbb{1}$, $Q_C(y) \geq v_\ell(v_\ell \backslash y \wedge \mathbb{1}) \geq v_\ell(\lambda_\ell \wedge \mathbb{1}) = v_\ell \lambda_\ell$. Summing over all $i \in I$, we get $Q_C(y) \geq z$. Together with (22), this shows (20). Since we also proved (23), this shows a fortiori (21).

Finally, if $\nu_C(y)$ is invertible, we see from (22) that $Q_C(y)(\nu_C(y))^{-1}$ is of the form $\bigoplus_{v \in C} v\lambda_v$ with $\bigoplus_{v \in C} \lambda_v = \mathbb{1}$, hence $Q_C(y)(\nu_C(y))^{-1}$ belongs to C. □

Example 2.5. In Figure 3, the convex C generated by three points (a, b, c) in $\bar{\mathbb{R}}^2_{\max}$ is displayed, together with its shadow and upper set. The cases of y belonging to $\mathrm{Up}(C)$ and of $y \notin \mathrm{Up}(C)$ are illustrated.

Remark 2.6. When $\mathcal{S} = \bar{\mathcal{K}}$ is a completed idempotent semifield, and C is complete and convex, then

$$\mathrm{Sh}(C) = \{x\lambda \mid \lambda \in \mathcal{K},\ \lambda \leq \mathbb{1},\ x \in C\} \ . \tag{24}$$

Indeed, let $\mathrm{Sh}'(C)$ denote the set in the right hand side of (24). The inclusion $\mathrm{Sh}'(C) \subset \mathrm{Sh}(C)$ is trivial. To show the other inclusion, take any $z \in \mathrm{Sh}(C)$, which can be written as a linear combination $z = \bigoplus_{\ell \in L} x_\ell \lambda_\ell$, for some $x_\ell \in C$, $\lambda_\ell \in \mathcal{S}$, $\lambda_\ell \leq \mathbb{1}$, with L a possibly infinite set. When $z = \mathbb{0}$, $z \in \mathrm{Sh}'(C)$ trivially. When $z \neq \mathbb{0}$, $\lambda_\ell \neq \mathbb{0}$ for some ℓ, so that $\mu := \bigoplus_{\ell \in L} \lambda_\ell \neq \mathbb{0}$, and since $\mu \leq \mathbb{1}$ and $\mathcal{S}$ is a completed idempotent semifield, μ is invertible. Writing $z = y\mu$, and observing that $y = \bigoplus_{\ell \in L} x_\ell \lambda_\ell \mu^{-1}$ belongs to C because C is complete and convex, we see that $z \in \mathrm{Sh}'(C)$. □

Example 2.7. To illustrate the previous results, consider the convex set $C \subset \bar{\mathbb{R}}^2_{\max}$ generated by the two points $(0, -\infty)$ and $(2, 3)$. Thus, C is the set of points of the form $(\max(\alpha, \beta + 2), \beta + 3)$, with $\max(\alpha, \beta) = 0$. Since C is generated by a finite number of points of $\bar{\mathbb{R}}^2_{\max}$, C is complete convex. The set C is the broken dark segment between the points $(0, -\infty)$ and $(2, 3)$, in Figure 4. In order to represent points with $-\infty$ coordinates, we use exponential coordinates in Figure 4, that is, the point $(z_1, z_2) \in \mathbb{R}^2_{\max}$ is represented by the point of the positive quadrant of coordinates $(\exp(z_1), \exp(z_2))$. Consider now $y = (1, -k)$, for any $k \geq 0$, and let us separate y from C using Corollary 2.2. Since $(0, -\infty) \leq y$, $y \in \mathrm{Up}(C)$, and we get from (21) that $\nu_C(y) = 0$. One also easily checks that $Q_C(y) = (0, -k)$. When

$k \neq +\infty$, the separating hyperplane H of (19) becomes:

$$H = \{x \in \bar{\mathbb{R}}^2_{\max} \mid \max(-1 + x_1, k + x_2, 0) = \max(x_1, k + x_2, 0)\} \ . \tag{25}$$

The point $y = (1, 0)$, together with $Q_C(y)$ and the separating hyperplane H (light grey zone) are depicted at the left of Figure 4. When $k = \infty$, is it easily checked that the separating hyperplane is the union of the half space $\bar{\mathbb{R}} \times (\mathbb{R} \cup \{\infty\})$, and of the interval $[-\infty, 0] \times \{-\infty\}$. Unlike in the case of a finite k, H is not closed for the usual topology, which implies that the max-plus linear forms which define H are *not* continuous for the usual topology.

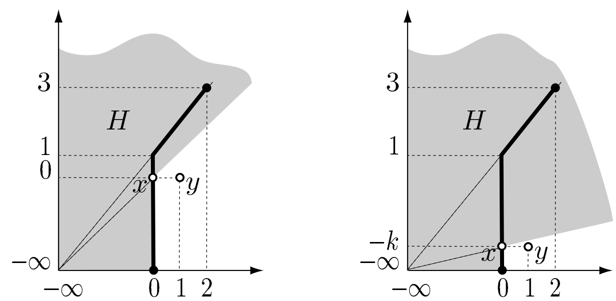

FIGURE 4. Separating a point from a convex set

2.4. **Closed convex sets in the order topology.** We next recall some basic facts about Birkhoff's order topology [Bir67, Ch. 10, § 9], and establish some properties of closed convex sets. See [GHK$^+$80, AS03] for more background on topologies on lattices and lattices ordered groups.

Recall that a nonempty ordered set D is *directed* if any finite subset of D has an upper bound in D, and that a nonempty ordered set F is *filtered* if any finite subset of F has a lower bound in F.

Definition 2.8. We say that a subset X of a conditionally complete ordered set S is *stable under directed sups* (resp. *stable under filtered infs*) if for all directed (resp filtered) subsets $D \subset X$ (resp. $F \subset X$) bounded from above (resp. below), $\bigvee D \in X$ (resp. $\bigwedge F \in X$).

When $S = \mathcal{K}^n$, where $\mathcal{K}$ is a conditionally complete idempotent semifield, the condition that F is bounded from below can be dispensed with, since any F is bounded from below by $\mathbb{0}$. Recall that a *net* with values in a conditionally complete ordered set S is a family $(x_\ell)_{\ell \in L} \subset S$ indexed by elements of a directed set $(L, \leq)$. We say that a net $(x_\ell)_{l \in L} \in S$ bounded from above and from below *order converges* to $x \in S$, if $x = \limsup_{\ell \in L} x_\ell = \liminf_{\ell \in L} x_\ell$, where $\limsup_{\ell \in L} x_\ell := \inf_{\ell \in L} \sup_{m \geq \ell} x_m$, and $\liminf_{\ell \in L} x_\ell := \sup_{\ell \in L} \inf_{m \geq \ell} x_m$. We say that $X \subset S$ is *order-closed* if for all nets $(x_\ell)_{\ell \in L} \subset X$ order converging to some $x \in S$, $x \in X$. The set $o(S)$ of order-closed subsets of S defines the Birkhoff's *order topology.* In particular, if D (resp. F) is a directed (resp. filtered) subset of X, $\{x\}_{x \in D}$ (resp. $\{x\}_{x \in F^{\mathrm{op}}}$) is a net which order converges to $\bigvee D$ (resp. $\bigwedge F$), so that any order closed set is stable under directed sups and filtered infs. We warn the reader that a net which is order convergent is convergent for the order topology, but that the converse need not hold, see [Bir67, Ch. 10, § 9]. However, both notions coincide when $S = \mathcal{K}^n$ if $\mathcal{K}$ is a conditionally complete semifield which is a continuous lattice [AS03]. When $S = \mathbb{R}^n_{\max}$, the order topology is the usual topology on $(\mathbb{R} \cup \{-\infty\})^n$.

The following result applies in particular to convex subsets of semimodules.

Proposition 2.9. *A subset $C \subset S$ stable under finite sups is closed for the order topology if and only if it is stable under directed sups and filtered infs.*

Proof. Assume that C is stable under directed sups and filtered infs, and let $\{x_\ell\}_{\ell\in L} \subset C$ denote a net order converging to $x \in S$. We have $x = \bigwedge_{\ell\in L} \bar{x}_\ell$, where $\bar{x}_\ell = \bigoplus_{m\geq \ell} x_m$. Let D_ℓ denote the set of finite subsets of $\{m \in L \mid m \geq \ell\}$, and for all $J \in D_\ell$, define $x_J = \bigoplus_{m\in J} x_m$. Since C is stable under finite sups, $x_J \in C$. Since $\{x_J \mid J \in D_\ell\}$ is directed and bounded from above, and since C is stable under directed sups, $\bar{x}_\ell = \bigoplus_{J\in D_\ell} \bigoplus_{m\in J} x_m = \bigoplus_{J\in D_\ell} x_J \in C$. Since $\{\bar{x}_\ell \mid \ell \in L\}$ is filtered and bounded from below, and since C is stable under filtered infs, $x = \bigwedge_{\ell\in L} \bar{x}_\ell \in C$, which shows that C is closed for the order topology. This shows the "if" part of the result. Conversely, if D (resp. F) is a directed (resp. filtered) subset of X, then $\{x\}_{x\in D}$ (resp. $\{x\}_{x\in F^{\mathrm{op}}}$) is a net which order converges to $\bigvee D$ (resp. $\bigwedge F$), so that any order closed set is stable under directed sups and filtered infs. □

We shall use repeatedly the following lemma in the sequel.

Lemma 2.10 (See. [Bir67, Ch. 13, Th. 26])**.** *If $\mathcal{K}$ is a conditionally complete semifield, if $x_\ell \in \mathcal{K}$ order converges to $x \in \mathcal{K}$, and $y_\ell \in \mathcal{K}$ order converges to $y \in \mathcal{K}$, then $x_\ell \wedge y_\ell$ order converges to $x \wedge y$, $x_\ell \oplus y_\ell$ order converges to $x \oplus y$, and $x_\ell y_\ell$ order converges to xy.*

In fact, the result of [Bir67] is stated only for elements of $\mathcal{K} \setminus \{\mathbb{0}\}$, but the extension to $\mathcal{K}$ is plain, since $x\mathbb{0} = \mathbb{0}x = x$, $x \oplus \mathbb{0} = \mathbb{0} \oplus x = x$ and $x \wedge \mathbb{0} = \mathbb{0} \wedge x = \mathbb{0}$ for all $x \in \mathcal{K}$. However, Lemma 2.10 does *not* extend to $\bar{\mathcal{K}}$: for instance, in $\bar{\mathbb{R}}_{\max}$, $(-\ell)_{\ell\in\mathbb{N}}$ order converges to $-\infty$, but $((+\infty) + (-\ell))_{\ell\in\mathbb{N}}$, which is the constant sequence with value $+\infty$, does not order converge to $(+\infty) + (-\infty) = -\infty$. This is precisely why the separating hyperplane provided by the universal separation theorem need not be closed, see Example 2.7 above.

Corollary 2.11. *If $v \in \mathcal{K}^n$, $w \in \mathcal{K}^n \setminus \{\mathbb{0}\}$, if $x_\ell \in \mathcal{K}^n$ order converges to $x \in \mathcal{K}^n$, and if $\lambda_\ell \in \mathcal{K}$ order converges to $\lambda \in \mathcal{K}$, then, $\langle v, x_\ell\rangle$ order converges to $\langle v, x\rangle$, $w \backslash x_\ell$ order converges to $w \backslash x$, and $v\lambda_\ell$ order converges to $v\lambda$.*

Proof. By (14), $\langle v, y\rangle = \bigoplus_{1\leq i\leq n} v_i y_i$, and by (9), $w \backslash y = \bigwedge_{i\in I} w_i^{-1} y_i$, where $I = \{1 \leq i \leq n \mid w_i \neq \mathbb{0}\} \neq \emptyset$, so the corollary follows from Lemma 2.10. □

We shall need the following basic property:

Lemma 2.12. *If C is a convex subset (resp. a subsemimodule) of $\mathcal{K}^n$, then, its closure for the order topology is a convex subset (resp. a subsemimodule) of $\mathcal{K}^n$.*

Proof. We derive this from Lemma 2.10 (the only unusual point is that the order convergence need not coincide with the convergence for the order topology). Assume that C is convex (the case when C is a semimodule is similar). Recall that if f is a continuous self-map of a topological space X, then $f(\mathrm{clo}(Y)) \subset \mathrm{clo} f(Y)$ holds for all $Y \subset X$, where $\mathrm{clo}(\cdot)$ denotes the closure of a subset of X. Fix $\alpha, \beta \in \mathcal{K}$ such that $\alpha \oplus \beta = \mathbb{1}$, and consider $\psi : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}^n$, $\psi(x, y) = x\alpha \oplus y\beta$. We claim that for all $x \in \mathcal{K}^n$, the map $\psi(x,\cdot)$ is continuous in the order topology. Indeed, let A denote a subset of $\mathcal{K}^n$ that is closed in the order topology, and let us show that the pre-image by $\psi(x,\cdot)$ of A, $A' = \{y \in \mathcal{K}^n \mid x\alpha \oplus y\beta \in A\}$, is also closed in the order topology. If $\{y_\ell\}_{\ell\in L}$ is any net in A' converging to some $y \in \mathcal{K}^n$, we have $x\alpha \oplus y_\ell\beta \in A$, for all $\ell \in L$, and it follows from Lemma 2.10 that $x\alpha \oplus y_\ell\beta$ order converges to $x\alpha \oplus y\beta$. Since A is closed in the order topology, $x\alpha \oplus y\beta \in A$, so $y \in A'$, which shows that A' is closed in the order topology. Thus, $\psi(x,\cdot)$ is continuous, and so $\psi(x, \mathrm{clo}(C)) \subset \mathrm{clo}(\psi(x, C))$. Since C is stable under convex combinations, $\psi(x, C) \subset C$, hence, $\psi(x, \mathrm{clo}(C)) \subset \mathrm{clo}(C)$. Pick now any $y \in \mathrm{clo}(C)$. Since $\psi(x, y) \in \mathrm{clo}(C)$, for all $x \in C$, and since $\psi(\cdot, y)$ is continuous, $\psi(\mathrm{clo}(C), y) \subset \mathrm{clo}(\psi(C, y)) \subset \mathrm{clo}(C)$. Since this holds for all $y \in \mathrm{clo}(C)$, we have shown that $\psi(\mathrm{clo}(C), \mathrm{clo}(C)) \subset \mathrm{clo}(C)$, i.e., $\mathrm{clo}(C)$ is stable under convex combinations. □

We conclude this section with properties which hold more generally in semimodules of functions. For all $C \subset \mathcal{K}^I$, we denote by $\bar{C} \subset \bar{\mathcal{K}}^I$ the set of arbitrary convex combinations of elements of C:

$$\bar{C} = \{\bigoplus_{\ell\in L} v_\ell\lambda_\ell \mid \{v_\ell\}_{\ell\in L} \subset C,\ \{\lambda_\ell\}_{\ell\in L} \subset \mathcal{K},\ \bigoplus_{\ell\in L} \lambda_\ell = \mathbb{1}\} \tag{26}$$

(L denotes an arbitrary - possibly infinite - index set).

Proposition 2.13. *If C is a convex subset of $\mathcal{K}^I$ which is closed in the order topology, then*

$$\bar{C} \cap \mathcal{K}^I = C\ . \tag{27}$$

Proof. Consider an element $v = \bigoplus_{\ell\in L} v_\ell\lambda_\ell \in \bar{C} \cap \mathcal{K}^I$, with $\bigoplus_{\ell\in L} \lambda_\ell = \mathbb{1}$. Assume, without loss of generality, that $\lambda_\ell \neq \mathbb{0}$, for all $\ell \in L$. Let D denote the set of finite subsets of L, and for all $J \in D$, let $v_J = \bigoplus_{\ell\in J} v_\ell\lambda_\ell$, and $\lambda_J = \bigoplus_{\ell\in J} \lambda_\ell$. By construction, the net $\{v_J\}_{J\in D}$ order converges to v, and the net $\{\lambda_J\}_{J\in D}$ order converges to $\mathbb{1}$. Hence, by Lemma 2.10, $v_J\lambda_J^{-1}$ order converges to v. But $v_J\lambda_J^{-1} = \bigoplus_{\ell\in J} v_\ell\lambda_\ell\lambda_J^{-1} \in C$, and since C is closed for the order topology, $v \in C$, which shows (27). □

When C is a semimodule, the condition that C is stable under filtered infs, which is implied by the condition that C is closed in the order topology, can be dispensed with.

Proposition 2.14. *If C is a subsemimodule of $\mathcal{K}^I$ which is stable under directed sups,* (27) *holds.*

Proof. Any element $v \in \bar{C}$ can be written as $v = \bigoplus_{\ell\in L} v_\ell$, for some $\{v_\ell\}_{\ell\in L} \subset C$. Setting $v_J = \bigoplus_{\ell\in J} v_\ell \in C$, we get $v = \bigoplus_{J\in D} v_J$, and we only need to know that C is stable under directed sups to conclude that $v \in C$. □

The following example shows that we cannot derive Proposition 2.14 from Proposition 2.13.

Example 2.15. The set $C = \{(-\infty, -\infty)\} \cup (\mathbb{R} \times \mathbb{R})$ is a subsemimodule of $\mathbb{R}^2_{\max}$, which is stable under directed sups, but not stable under filtered infs (for instance $\bigwedge\{(0, -\ell) \mid \ell \in \mathbb{N}\} = (0, -\infty) \notin C$), and hence not closed in the order topology.

3. Separation theorems for closed convex sets

We saw in Example 2.7 that, when $\mathcal{K} = \mathbb{R}_{\max}$, the separating set (19) given by the universal separation theorem need not be closed for the usual topology. In this section, we refine the universal separation theorem in order to separate a point from a *closed* convex set by a *closed* hyperplane.

From now on, we assume that $\mathcal{K}$ is a conditionally complete idempotent semifield, whose completed semiring is denoted by $\bar{\mathcal{K}}$.

3.1. Separation of closed convex subsets of $\mathcal{K}^I$.

As a preparation for the main result of §3 (Theorem 3.14 below), we derive from Corollary 2.2 a separation result for order closed convex sets C and elements $y \in X \setminus C$ of the semimodule of functions $X = \mathcal{K}^I$, satisfying an archimedean condition. This archimedean condition will be suppressed in Theorem 3.14, assuming that I is finite.

Definition 3.1. We call *affine hyperplane* of $\mathcal{K}^I$ a subset of $\mathcal{K}^I$ of the form

$$H = \{v \in \mathcal{K}^I \mid \langle w', x\rangle \oplus d' = \langle w'', x\rangle \oplus d''\} \ , \tag{28}$$

with $w', w'' \in \mathcal{K}^I$, and $d', d' \in \mathcal{K}$. We shall say that H is a *linear hyperplane* if $d' = d'' = \mathbb{0}$.

(When I is infinite, $\langle w', x\rangle$ and $\langle w'', x\rangle$ may be equal to $\top$.)

Remark 3.2. We have already encountered "hyperplanes" of $\mathcal{K}^I$ of the above form. Indeed, $H \cap \mathcal{K}^I$, with H of (19), is of the form (28), with

$$w' = {}^{-}y,\ d' = \mathbb{1},\ w'' = {}^{-}Q_C(y),\ d'' = {}^{-}\nu_C(y) \ . \tag{29}$$

The main point in Definition 3.1 is the requirements that $w', w'' \in \mathcal{K}^I$, and $d', d'' \in \mathcal{K}$ which need not be satisfied in (29); indeed, for $y = (y_i) \in \mathcal{K}^I$ having a coordinate $y_{i_0} = \mathbb{0}$, by (15), we have ${}^{-}y_i = \top\bar{\mathcal{K}}$, so that ${}^{-}y \notin \mathcal{K}^I$ (see e.g. Example 2.7).

Given $y \in \mathcal{K}^I \setminus C$, the question is whether we can find an affine hyperplane of $\mathcal{K}^I$ containing C and not y. We shall need the following *Archimedean type assumption* on C and y:

$$(A): \qquad \forall v \in C, \exists \lambda \in \mathcal{K} \setminus \{\mathbb{0}\}, v\lambda \leq y \ .$$

For all $y \in \mathcal{K}^I$ and $C \subset \mathcal{K}^n$, define

$$\operatorname{supp} y = \{i \in I \mid y_i \neq \mathbb{0}\} \ , \qquad \operatorname{supp} C = \bigcup_{v \in C} \operatorname{supp} v \ .$$

One readily checks that $y\lambda \leq y'$ for some $\lambda \in \mathcal{K} \setminus \{\mathbb{0}\}$ implies that $\operatorname{supp} y \subset \operatorname{supp} y'$, and that when I is finite, the converse implication holds (indeed, if $\operatorname{supp} y \subset \operatorname{supp} y'$, take any $\lambda \in \mathcal{K}$ smaller than $y \backslash y' = \wedge_{i \in I} y_i \backslash y'_i$, a quantity which is in $\bar{\mathcal{K}} \setminus \{\varepsilon\}$ when I is finite). Thus, Assumption (A) implies that

$$\operatorname{supp} y \supset \operatorname{supp} C \ , \tag{30}$$

and it is equivalent to (30) when I is finite.

Proposition 3.3. *Let C be a convex subset of $\mathcal{K}^I$, and $y \in \mathcal{K}^I \setminus C$. Assume that C is closed for the order topology of $\mathcal{K}^I$, and that Assumption (A) is satisfied. Then, there is an affine hyperplane of $\mathcal{K}^I$ which contains C and not y.*

Remark 3.4. The separating hyperplanes built in the proof of Proposition 3.3 can be written as (28), with $w' \geq w''$ and $d' \geq d''$, so that, by the same argument as in Remark 2.3, H in (28) may be rewritten as $H = \{v \in \mathcal{K}^I \mid \langle w', x\rangle \oplus d' \leq \langle w'', x\rangle \oplus d''\}$.

Proof. First, we can assume that $\operatorname{supp} y \subset \operatorname{supp} C$, which, by (30), means that

$$\operatorname{supp} y = \operatorname{supp} C \enspace . \tag{31}$$

Otherwise, there is an index $i \in I$ such that $y_i \neq \mathbb{0}$ and $v_i = \mathbb{0}$, for all $v \in C$, so that the hyperplane of equation $v_i = \mathbb{0}$ contains C and not y.

We can also assume that

$$\operatorname{supp} C = I \enspace . \tag{32}$$

Indeed, if $\operatorname{supp} C \neq I$, we set $J := \operatorname{supp} C$, and consider the restriction map $r : \mathcal{K}^I \to \mathcal{K}^J$, which sends a vector $x \in \mathcal{K}^I$ to $r(x) = (x_j)_{j \in J}$. We have $\operatorname{supp} r(y) \subset \operatorname{supp} r(C) = J$. Assuming that the theorem is proved when (32) holds, we get vectors $w', w'' \in \mathcal{K}^J$ and scalars $d', d'' \in \mathcal{K}$ such that the affine hyperplane $H = \{x \in \mathcal{K}^J \mid \langle w', x\rangle \oplus d' = \langle w'', x\rangle \oplus d''\}$ contains $r(C)$ and not $r(y)$. Let $\hat{w}'$ and $\hat{w}''$ denote the vectors obtained by completing w' and w'' by zeros. Then, the hyperplane $\hat{H} = \{x \in \mathcal{K}^I \mid \langle \hat{w}', x\rangle \oplus d' = \langle \hat{w}'', x\rangle \oplus d''\}$ contains C and not y.

It remains to show Proposition 3.3 when the equalities (31), (32) hold. Define the complete convex set $\bar{C}$ as in (26). It follows from (27) that $y \notin \bar{C}$. Therefore, defining $Q_{\bar{C}}(y)$ and $\nu_{\bar{C}}(y)$ as in (18), with C replaced by $\bar{C}$, we get that the set H of (19), where C is replaced by $\bar{C}$, contains C and not y. By (31) and (32), we have $\operatorname{supp} y = I$, so ${}^-y \in \mathcal{K}^I$. Also, by (20) and (27), $Q_{\bar{C}}(y) \in \mathcal{K}^I$. Moreover $\nu_{\bar{C}}(y) \in \mathcal{K}$ ($\mathcal{K}$ is conditionally complete and $\nu_{\bar{C}}(y)$ is the sup of a family of elements bounded from above by the unit). If

$$(Q_{\bar{C}}(y))_i \neq \mathbb{0},\ \forall i \in I, \text{ and } \nu_{\bar{C}}(y) \neq \mathbb{0} \enspace , \tag{33}$$

we will have ${}^-Q_{\bar{C}}(y) \in \mathcal{K}^I$, and ${}^-\nu_{\bar{C}}(y) \in \mathcal{K}$, and the set $H \cap \mathcal{K}^I$, where H is as in (19), will be an affine hyperplane of $\mathcal{K}^I$. In order to show (33), take any $i \in I$. Since the equalities (31) and (32) hold, we can find $v \in C$ such that $v_i \neq \mathbb{0}$, and thanks to Assumption (A), $v\lambda \leq y$, for some $\lambda \in \mathcal{K} \setminus \{\mathbb{0}\}$. Hence,

$$\nu_{\bar{C}}(y) \geq v \backslash y \wedge \mathbb{1} \geq v \backslash (v\lambda) \wedge \mathbb{1} \geq \lambda \wedge \mathbb{1} > \mathbb{0} \enspace ,$$

and

$$(Q_{\bar{C}}(y))_i \geq v_i(v \backslash y \wedge \mathbb{1}) \geq v_i(\lambda \wedge \mathbb{1}) > \mathbb{0} \enspace ,$$

which shows (33). □

When C is a semimodule, the condition that C is stable under filtered infs (which is implied by the condition that C is order closed) can be dispensed with.

Proposition 3.5. *Let C be a subsemimodule of $\mathcal{K}^I$, and $y \in \mathcal{K}^I \setminus C$. Assume that C is stable under directed sups, and that Assumption (A) is satisfied. Then, there is a linear hyperplane of $\mathcal{K}^I$ which contains C and not y.*

Proof. We reproduce the proof of Proposition 3.3, using directly (16), where $V = \bar{C}$, and noting that, by Proposition 2.14, (27) holds as soon as C is stable under directed sups, when C is a semimodule. □

In Proposition 3.3, we required the convex set to be order closed, but the separating sets, namely the affine hyperplanes of $\mathcal{K}^I$, where I is infinite, need not be order closed, as shown by the following counter-example.

Example 3.6. Let $I = \mathbb{N}$, $\mathcal{K} = \mathbb{R}_{\max}$, and let us separate $y = (0,1,0,1,0,1\ldots)$ from the convex set $C = \{(0,0,0,\ldots)\}$ using Proposition 3.3. We obtain the affine hyperplane $H = \{x \in \mathbb{R}_{\max}^{\mathbb{N}} \mid a(x) \oplus 0 = b(x) \oplus 0\}$, where $a(x) = x_0 \oplus (-1)x_1 \oplus x_2 \oplus (-1)x_3 \oplus \cdots$, and $b(x) = x_0 \oplus x_1 \oplus x_2 \oplus \cdots$. Consider the decreasing sequence $y^\ell \in \mathbb{R}_{\max}^{\mathbb{N}}$, such that $y^\ell_{2i+1} = 2$, for all $i \in \mathbb{N}$, and $y^\ell_{2i} = 1$, for $i \leq \ell$, and $y^\ell_{2i} = 2$, for $i > \ell$, so that $y^\ell \in H$ for all ℓ. We have $\inf_\ell y^\ell = y$, where $y_{2i+1} = 2$ for all $i \in \mathbb{N}$, and $y_{2i} = 1$, for all $i \in \mathbb{N}$. Since $y \notin H$, H is not stable under filtered infs.

Of course, this pathology vanishes in the finite dimensional case.

Proposition 3.7. *Affine hyperplanes of $\mathcal{K}^n$ are closed in the order topology.*

Proof. This follows readily from Lemma 2.10. □

The following example shows that the archimedean assumption is useful in Proposition 3.5.

Example 3.8. Consider the semimodule $C = \{(-\infty,-\infty)\} \cup \{(x_1,x_2) \in \mathbb{R}\times\mathbb{R} \mid x_1 \geq x_2\} \subset \mathbb{R}_{\max}^2$, which is stable under directed sups, and consider the point $y = (0,1) \notin C$, with $\operatorname{supp} y = \operatorname{supp} C = \{1,2\}$, so that Assumption (A) is satisfied. The proof of Theorem 3.5 allows us to separate y from C by the linear hyperplane:

$$H = \{(x_1,x_2) \in \mathbb{R}_{\max}^2 \mid x_1 \oplus (-1)x_2 = x_1 \oplus x_2\} \ . \tag{34}$$

However, consider now $y = (0,-\infty) \notin C$, which does not satisfy Assumption (A). We cannot separate y from C by a linear (or affine) hyperplane, because such an hyperplane would be closed in the order (=usual) topology of $\mathbb{R}_{\max}^2$ (by Proposition 3.7) whereas y belongs to the closure of C in this topology. Thus Assumption (A) cannot be ommited in Proposition 3.3.

3.2. **Projectors onto closed semimodules of $\mathcal{K}^n$.** In order to show that in the finite dimensional case, Assumption (A) is not needed in Proposition 3.3, we establish some continuity property for projectors onto closed semimodules of $\mathcal{K}^n$.

If V is a subsemimodule of $\mathcal{K}^n$, we define $\bar{V} \subset \bar{\mathcal{K}}^n$ as in (26) (the condition $\bigoplus_{\ell\in L} \lambda_\ell = \mathbb{1}$ can be dispensed with, since V is a semimodule), together with the projector $P_{\bar{V}} : \bar{\mathcal{K}}^n \to \bar{\mathcal{K}}^n$,

$$P_{\bar{V}}(x) = \top\{v \in \bar{V} \mid v \leq x\} \ . \tag{35}$$

Since $\{v\}_{v\in V}$ is a generating family of the complete semimodule $\bar{V}$, it follows from (12) that

$$P_{\bar{V}}(x) = \bigoplus_{v\in V} v(v \backslash x) \ . \tag{36}$$

Proposition 3.9. *If V is a subsemimodule of $\mathcal{K}^n$, that is stable under directed sups, then the projector $P_{\bar{V}}$ from $\bar{\mathcal{K}}^n$ onto $\bar{V}$ admits a restriction P_V from $\mathcal{K}^n$ to V.*

Proof. If $y \in \mathcal{K}^n$, $P_{\bar{V}}(y) \leq y$ also belongs to $\mathcal{K}^n$, so that by (27), $P_{\bar{V}}(y) \in \bar{V} \cap \mathcal{K}^n = V$. □

Definition 3.10. We say that a map f from S to an ordered set T *preserves directed sups* (resp. *preserves filtered infs*) if $f(\bigvee D) = \bigvee f(D)$ (resp. $f(\bigwedge F) = \bigwedge f(F)$ for all directed subsets $D \subset S$ bounded from above (resp. for all filtered subsets $F \subset S$ bounded from below).

Proposition 3.11. *If V is a subsemimodule of $\mathcal{K}^n$ stable under directed sups and filtered infs, then P_V preserves directed sups and filtered infs.*

Proof. Let F denote a filtered subset of V, and $x = \bigwedge F$. Then, by $\bigwedge F \geq P_V(\bigwedge F)$, and since P_V is isotone, we have

$$P_V(x) = P_V(\bigwedge F) \geq P_V(\bigwedge P_V(F)) \ . \tag{37}$$

Furthermore, since P_V is isotone, $P_V(F)$ is filtered (indeed, if F' is any finite subset of $P_V(F)$, we can write $F' = P_V(F'')$ for some finite subset $F'' \subset F$; since F is filtered, F'' has a lower bound $t \in F$, and since P_V is isotone, $P_V(t) \in P_V(F)$ is a lower bound of $F' = P_V(F'')$, which shows that $P_V(F)$ is filtered). Hence, $\bigwedge P_V(F) \in V$ because V is stable under filtered infs. Since P_V fixes V, $P_V(\bigwedge P_V(F)) = \bigwedge P_V(F)$, and we get from (37), $P_V(\bigwedge F) \geq \bigwedge P_V(F)$. The reverse inequality is an immediate consequence of the isotony of P_V.

Consider now a directed subset $D \subset V$ bounded from above, and $x = \bigvee D = \bigoplus_{y\in D} y$. We first show that for all $v \in \mathcal{K}^n$,

$$\bigoplus_{y\in D} v(v \backslash y) = v(v \backslash \bigoplus_{y\in D} y) \ . \tag{38}$$

We shall assume that $v \neq \mathbb{0}$ (otherwise, the equality is trivial). Since D is directed, the net $\{y\}_{y\in D}$ order converges to $\bigoplus_{y\in D} y$, and by Corollary 2.11, this implies that $\{v(v \backslash y)\}_{y\in D}$ order converges to $v(v \backslash \bigoplus_{y\in D} y)$. Since $y \mapsto v(v \backslash y)$ is isotone, $\{v(v \backslash y)\}_{y\in D}$ order converges to its sup. (Indeed, let φ denote an isotone map from D to a conditionally complete ordered set, such that $\{\varphi(y)\}_{y\in D}$ is bounded from above, and let us show more generally that $\{\varphi(y)\}_{y\in D}$ order converges to its sup. Observe that $\sup_{z\geq y} \varphi(z) = \bigvee \varphi(D)$ is independent of $y \in D$ because D is directed and φ is isotone. Then, $\limsup_{y\in D} \varphi(y) = \inf_{y\in D} \sup_{y'\geq y} \varphi(y') = \bigvee \varphi(D)$. Also, since φ is isotone, $\liminf_{y\in D} \varphi(y) = \sup_{y\in D} \inf_{y'\geq y} \varphi(y') = \sup_{y\in D} \varphi(y) = \bigvee \varphi(D)$, which shows that $\{\varphi(y)\}_{y\in D}$ order converges to its sup.) So, (38) is proved.

Using (36), we get

$$\begin{aligned}\bigoplus_{y\in D} P_V(y) = \bigoplus_{y\in D}\bigoplus_{v\in V} v(v \backslash y) = \bigoplus_{v\in V}\bigoplus_{y\in D} v(v \backslash y) &= \bigoplus_{v\in V} v\big(v \backslash (\bigoplus_{y\in D} y)\big) \quad \text{(by (38))}\\ &= P_V(x) \ .\end{aligned}$$

□

Remark 3.12. Proposition 3.11 does not extend to semimodules of the form $\mathcal{K}^I$, where I is an infinite set. Indeed, take $I = \mathbb{N}$, $\mathcal{K} = \mathbb{R}_{\max}$, and let V denote the semimodule spanned by the vector $v = (0, 0, 0, \ldots)$. For all $x = (x_0, x_1, x_2, \ldots) \in \mathbb{R}_{\max}^{\mathbb{N}}$, we have $P_V(x) = (\lambda(x), \lambda(x), \ldots)$, where $\lambda(x) := \bigwedge_{i\in\mathbb{N}} x_i$. Consider now the sequence $y^k \in \mathbb{R}_{\max}^{\mathbb{N}}$, such that $y_i^k = 0$ if $k \leq i$, and $y_i^k = -1$, otherwise. Then, $\{y^k\}_{k\in\mathbb{N}}$ is a non-decreasing sequence with supremum v. We have $\lambda(y^k) = -1$, but $\lambda(v) = 0$, which shows that P_V does not preserve directed sups.

The proof of Theorem 3.14 will rely on the following corollary of Proposition 3.11.

Corollary 3.13. *If V is a subsemimodule of $\mathcal{K}^n$ stable under directed sups and filtered infs, and if $y \in \mathcal{K}^n \setminus V$, then, there is a vector $z \geq y$ with coordinates in $\mathcal{K} \setminus \{\mathbb{0}\}$, such that:*

$$y \not\leq P_V(z) \ . \tag{39}$$

Proof. Let Z denote the set of vectors $z \geq y$ with coordinates in $\mathcal{K} \setminus \{\mathbb{0}\}$. Let us assume by contradiction that

$$y \leq P_V(z), \quad \forall z \in Z \ . \tag{40}$$

Since by Proposition 3.11, P_V preserves filtered infs, we get from (40):

$$y \leq \bigwedge_{z \in Z} P_V(z) = P_V(\bigwedge_{z \in Z} z) = P_V(y) \ .$$

Since $y \geq P_V(y)$ holds trivially, $y = P_V(y)$, hence, $y \in V$, a contradiction. □

3.3. **Separation theorem for closed convex subsets of $\mathcal{K}^n$.** The following finite dimensional separation theorem extends an earlier result of Zimmermann [Zim77]. Recall that when $\mathcal{K} = \mathbb{R}_{\max}$, the order topology on $\mathcal{K}^n$ is the usual topology on $\mathbb{R}^n_{\max}$.

Theorem 3.14. *Let C denote a convex subset of $\mathcal{K}^n$ that is closed for the order topology of $\mathcal{K}^n$, and let $y \notin C$. Then, there exists an affine hyperplane containing C and not x.*

We shall need the following lemma:

Lemma 3.15. *If C is a semimodule, and if* $\operatorname{supp} y = \operatorname{supp} C$*, then* $\operatorname{supp} P_C(y) = \operatorname{supp} y$.

Proof. Since $P_C(y) \leq y$, $\operatorname{supp} P_C(y) \subset \operatorname{supp} y$. Conversely, pick any $i \in \operatorname{supp} y$. Since $\operatorname{supp} y = \operatorname{supp} C$, we can find $v \in C$ such that $\operatorname{supp} v \subset \operatorname{supp} y$ and $i \in \operatorname{supp} v$. Then, by (9), $v \backslash y = \bigwedge_{j \in \operatorname{supp} v} v_j^{-1} y_j \neq \mathbb{0}$, and since $P_C(y) \geq v(v \backslash y)$, $(P_C(y))_i \neq \mathbb{0}$, which shows that $\operatorname{supp} y \subset \operatorname{supp} P_C(y)$. □

We showed in the first part of the proof of Proposition 3.3 that we can always assume that

$$\operatorname{supp} y \subset \operatorname{supp} C = \{1, \ldots, n\} \ . \tag{41}$$

We next prove Theorem 3.14 in the special case where C is a semimodule, and then, we shall derive Theorem 3.14, in general.

Proof of Theorem 3.14 when C is a semimodule. The proof relies on the following perturbation argument. Pick a vector $z \geq y$ with coordinates in $\mathcal{K} \setminus \{\mathbb{0}\}$, (hence, by (41), $\operatorname{supp} z = \{1, \ldots, n\} = \operatorname{supp} C$), and define, as in (13),(16):

$$H(z) = \{x \in \mathcal{K}^n \mid x \backslash z = x \backslash P_C(z)\} = \{x \in \mathcal{K}^n \mid \langle {}^-z, x\rangle = \langle {}^-P_C(z), x\rangle\} \ .$$

We will show that $H(z)$ is a (linear) hyperplane, and that one can choose the above z so that $H(z)$ contains C and not y.

It follows from $\operatorname{supp} z = \operatorname{supp} C$ and Lemma 3.15 that $\operatorname{supp} P_C(z) = \operatorname{supp} z = \{1, \ldots, n\}$. Since ${}^-u \in \mathcal{K}^n$ for all vectors u of $\mathcal{K}^n$ with coordinates different from $\mathbb{0}$, ${}^-z$ and ${}^-(P_C(z))$ belong to $\mathcal{K}^n$, which shows that $H(z)$ is an hyperplane.

By Theorem 2.1, $H(z)$ contains C. Let us check that:

$$x \in H(z) \implies P_C(z) \geq x \ . \tag{42}$$

Recall the classical residuation identity

$$x(x \backslash x) = x \tag{43}$$

(this can be shown by applying the first identity in (6) to the map $f : \mathcal{K} \to \mathcal{K}^n, f(\lambda) = x\lambda$). If $x \in H(z)$, we have $x \backslash z = x \backslash P_C(z)$, and by (8), $P_C(z) \geq$

$x(x \backslash z)$. Using $z \geq x$ and (43), we get $P_C(z) \geq x(x \backslash z) \geq x(x \backslash x) = x$, which shows (42).

By Corollary 3.13, there is a vector $z \geq y$ with entries in $\mathcal{K} \setminus \{\mathbb{0}\}$ such that $y \not\leq P_C(z)$, and by (42), the hyperplane $H(z)$ associated to this z contains C and not y. □

Associate to a convex set $C \subset \mathcal{K}^n$ the semimodule:

$$V_C := \{(x\lambda, \lambda) \mid x \in C,\ \lambda \in \mathcal{K}\} \subset \mathcal{K}^{n+1} \ .$$

We denote by $\mathrm{clo}(V_C)$ the closure of V_C for the order topology of $\mathcal{K}^{n+1}$. We shall need the following:

Lemma 3.16. *If C is a convex subset of $\mathcal{K}^n$ closed for the order topology,*

$$\mathrm{clo}(V_C) \subset V_C \cup (\mathcal{K}^n \times \{\mathbb{0}\}) \ . \tag{44}$$

Proof. It suffices to show that $V_C \cup (\mathcal{K}^n \times \{\mathbb{0}\})$ is closed in $\mathcal{K}^{n+1}$ for the order topology. Take a net $\{(z_\ell, \lambda_\ell)\}_{\ell \in L} \subset V_C \cup (\mathcal{K}^n \times \{\mathbb{0}\})$, with $z_\ell \in \mathcal{K}^n, \lambda_\ell \in \mathcal{K}$, order converging to some (z, λ), with $z \in \mathcal{K}^n, \lambda \in \mathcal{K}$. We only need to show that if $\lambda \neq \mathbb{0}$, $(z, \lambda) \in V_C$. Since $\lambda \neq \mathbb{0}$, replacing L by a set of the form $\{\ell \in L \mid \ell \geq \ell_0\}$, we may assume that $\lambda_\ell \neq \mathbb{0}$, for all $\ell \in L$. Then, $(z_\ell, \lambda_\ell) \in V_C$, which implies that $z_\ell \lambda_\ell^{-1} \in C$. Since z_ℓ order converges to z, and λ_ℓ^{-1} order converges to λ^{-1}, by Lemma 2.10, $z_\ell \lambda_\ell^{-1}$ order-converges to $z\lambda^{-1}$. Since, by our assumption, C is closed for the order topology, $z\lambda^{-1} \in C$, which shows that $(z, \lambda) \in V_C$. So, $V_C \cup (\mathcal{K}^n \times \{\mathbb{0}\})$ is closed for the order topology. □

Derivation of the general case of Theorem 3.14. We now conclude the proof of Theorem 3.14. Let us take $y \in \mathcal{K}^n \setminus C$. We note that by (44), $(y, \mathbb{1}) \notin \mathrm{clo}(V_C)$. Applying Theorem 3.14, which is already proved in the case of closed semimodules, to $\mathrm{clo}(V_C)$, which is a semimodule thanks to Lemma 2.12, we get a linear hyperplane $H = \{\bar{x} \in \mathcal{K}^{n+1} \mid \langle w', \bar{x} \rangle = \langle w'', \bar{x} \rangle\}$, where $w', w'' \in \mathcal{K}^{n+1}$, such that

$$x \in C \implies (x, \mathbb{1}) \in H, \qquad \text{and } (y, \mathbb{1}) \notin H \ . \tag{45}$$

Introducing $z' = (w'_i)_{1 \leq i \leq n}$, and $z'' = (w''_i)_{1 \leq i \leq n}$, we see from (45) that the affine hyperplane:

$$\{x \in \mathcal{K}^n \mid \langle z', x \rangle \oplus w'_{n+1} = \langle z'', x \rangle \oplus w''_{n+1}\}$$

contains C and not y. □

Remark 3.17. We needed to introduce the closure $\mathrm{clo}(V_C)$ in the proof of Theorem 3.14 because V_C need not be closed when C is closed and convex. Indeed, consider $C = \{x \in \mathbb{R}_{\max} \mid x \geq 0\}$. We have $V_C = \{(\mathbb{0}, \mathbb{0})\} \cup \{(x\lambda, \lambda) \mid x \geq 0, \lambda \in \mathbb{R}\}$ and $\mathrm{clo}(V_C) = (\mathbb{R}_{\max} \times \{\mathbb{0}\}) \cup \{(x\lambda, \lambda) \mid x \geq 0, \lambda \in \mathbb{R}\}$.

Example 3.18. When applied to Example 2.7, the proof of Theorem 3.14 shows that for k large enough, the hyperplane H in (25) separates the point $(0, -\infty)$ from the convex set of Figure 4. The method of [SS92], which requires that the vector to separate from a convex should have invertible entries in order to apply a normalization argument, does not apply to this case.

4. Convex functions over idempotent semifields

We say that a map $f : \mathcal{K}^n \to \bar{\mathcal{K}}$ is *convex* if its epigraph is convex. By [Zim79a, Theorem 1], f is convex if, and only if,

$$(x, y \in \mathcal{K}^n, \alpha, \beta \in \mathcal{K}, \alpha \oplus \beta = \mathbb{1}) \Rightarrow f(x\alpha \oplus y\beta) \leq f(x)\alpha \oplus f(y)\beta \ . \tag{46}$$

Additionally, by [Zim79a, Theorem 2], the (lower) level sets

$$S_t(f) = \{x \in X \mid f(x) \leq t\} \qquad (t \in \bar{\mathcal{K}}) \tag{47}$$

of f are convex subsets of $\mathcal{K}^n$. When $\mathcal{K} = \mathbb{R}_{\max}$, we say that f is *max-plus convex*. Convex functions may of course be defined from an arbitrary $\mathcal{K}$-semimodule X to $\bar{\mathcal{K}}$: we limit our attention to $X = \mathcal{K}^n$ since the proof of the main result below relies on Theorem 3.14 which is stated for $\mathcal{K}^n$.

The following immediate proposition shows that the set of convex functions is a complete subsemimodule of the complete semimodule of functions $\mathcal{K}^n \to \bar{\mathcal{K}}$:

Proposition 4.1. *The set of all convex functions is stable under (arbitrary) pointwise sup, and under multiplication by a scalar (in $\bar{\mathcal{K}}$).* □

We defined in the introduction U*-convex* functions and sets, when $U \subset \bar{\mathbb{R}}^X$, see Equations (1) and (2). When more generally $U \subset \bar{\mathcal{K}}^X$, we still define U*-convex functions* by (1), and extend (2) by saying that a subset $C \subset \mathcal{K}^n$ is U*-convex* if for all $y \in X \setminus C$, we can find a map $u \in U$ such that

$$u(y) \not\leq \sup_{x \in C} u(x) \ . \tag{48}$$

Proposition 4.2. *A subset $C \subset X$ is U-convex if, and only if, it is an intersection of (lower) level sets of maps in U.*

Proof. Assume that C is an intersection of (lower) level sets of maps in U, that is, $C = \bigcap_{\ell \in L} S_{t_\ell}(u_\ell)$, where $\{u_\ell\}_{\ell \in L} \subset U$, $\{t_\ell\}_{\ell \in L} \subset \bar{\mathcal{K}}$, and L is a possibly infinite set. If $y \in X \setminus C$, $y \notin S_{t_\ell}(u_\ell)$, for some $\ell \in L$, so that $u_\ell(y) \not\leq t_\ell$. Since $C \subset S_{t_\ell}(u_\ell)$, we have $\sup_{x \in C} u_\ell(x) \leq t_\ell$. We deduce that $u_\ell(y) \not\leq \sup_{x \in C} u_\ell(x)$. Hence, C is U-convex.

Conversely, assume that C is U-convex, and let C' denote the intersection of the sets $S_t(u)$, with $u \in U$ and $t \in \bar{\mathcal{K}}$, in which C is contained. Trivially, $C \subset C'$. If $y \in X \setminus C$, we can find $u \in U$ satisfying (48). Let $t := \sup_{x \in C} u(x)$. Then, $C \subset S_t(u)$, and $y \notin S_t(u)$, so that $y \notin C'$. This shows that $X \setminus C \subset X \setminus C'$. Thus, $C = C'$. □

The set U of elementary functions which will prove relevant for our convex functions is the following.

Definition 4.3. We say that $u : \mathcal{K}^n \to \mathcal{K}$ is *affine* if $u(x) = \langle w, x \rangle \oplus d$, for some $w \in \mathcal{K}^n$ and $d \in \mathcal{K}$. We say that u is a *difference of affine functions* if

$$u(x) = (\langle w', x \rangle \oplus d') \ominus (\langle w'', x \rangle \oplus d'') \ , \tag{49}$$

where w', w'' belong to $\mathcal{K}^n$ and d', d'' to $\mathcal{K}$.

We illustrate in Table 1 below, and in Figure 2 of §1, the various shapes taken by differences of affine functions, when $n = 1$, and $\mathcal{K} = \mathbb{R}_{\max}$. For simplicity, a generic function in this class is denoted

$$y = (ax \oplus b) \ominus (cx \oplus d)$$

with $a, b, c, d \in \mathcal{K}$. Table 1 enumerates the four possible situations according to the comparisons of a with c and b with d. Figure 2 shows the corresponding plots.

	$a \leq c$	$a > c$
$b \leq d$	$y = \mathbb{0}, \quad \forall x$	$y = \begin{cases} \mathbb{0} & \text{if } x \leq a \backslash d, \\ ax & \text{otherwise.} \end{cases}$
$b > d$	$y = \begin{cases} b & \text{if } x < b \backslash c, \\ \mathbb{0} & \text{otherwise.} \end{cases}$	$y = ax \oplus b$

TABLE 1. The four generic differences of affine functions over $\mathbb{R}_{\max}$

Proposition 4.4. *The (lower) level sets of differences of affine functions are precisely the affine hyperplanes of the form:*

$$\{x \in \mathcal{K}^n \mid \langle w'', x\rangle \oplus d'' \geq \langle w', x\rangle \oplus d'\} \ , \tag{50}$$

where w', w'' belong to $\mathcal{K}^n$ and d', d'' to $\mathcal{K}$.

Proof. This is an immediate consequence of (11). □

Remark 4.5. The inequality in (50) is equivalent to the equality $\langle w'', x\rangle \oplus d'' = \langle w' \oplus w'', x\rangle \oplus d' \oplus d''$, which justifies the term "affine hyperplane".

We shall say that $f : \mathcal{K}^n \to \bar{\mathcal{K}}$ is *lower semi-continuous* if all (lower) level sets of f are closed in the order topology of $\mathcal{K}^n$.

Proposition 4.6. *Every difference of affine functions is convex and lower semi-continuous.*

Proof. If u is a difference of affine functions, by Proposition 4.4, the (lower) level sets of u, are affine hyperplanes, which are closed by Proposition 3.7, so u is lower semi-continuous.

As mentioned earlier, $u(\cdot) = (\langle w', \cdot\rangle \oplus d') \ominus (\langle w'', \cdot\rangle \oplus d'')$ is convex if and only if its epigraph is convex. So, we consider two points (x_1, λ_1) and (x_2, λ_2) in the epigraph of u, namely,

$$\lambda_1 \geq (\langle w', x_1\rangle \oplus d') \ominus (\langle w'', x_1\rangle \oplus d'') \ ,$$
$$\lambda_2 \geq (\langle w', x_2\rangle \oplus d') \ominus (\langle w'', x_2\rangle \oplus d'') \ ,$$

which, by (11), is equivalent to

$$\lambda_1 \oplus \langle w'', x_1\rangle \oplus d'' \geq \langle w', x_1\rangle \oplus d' \ ,$$
$$\lambda_2 \oplus \langle w'', x_2\rangle \oplus d'' \geq \langle w', x_2\rangle \oplus d' \ .$$

Let α and β in $\mathcal{K}$ be such that $\alpha \oplus \beta = \mathbb{1}$. From the previous inequalities, we derive

$$\lambda_1\alpha \oplus \lambda_2\beta \oplus \langle w'', x_1\alpha \oplus x_2\beta\rangle \oplus d'' \geq \langle w', x_1\alpha \oplus x_2\beta\rangle \oplus d' \ ,$$

which, by (11), is equivalent to

$$\lambda_1\alpha \oplus \lambda_2\beta \geq u(x_1\alpha \oplus x_2\beta) \ .$$

We have proved that $(x_1\alpha \oplus x_2\beta, \lambda_1\alpha \oplus \lambda_2\beta)$ belongs to the epigraph of u. Thus, u is convex. □

Corollary 4.7. *Let $C \subset \mathcal{K}^n$. The following assertions are equivalent:*

(1) *C is a convex subset of $\mathcal{K}^n$, and it is closed in the order topology;*
(2) *C is U-convex, where U denotes the set of differences of affine functions $\mathcal{K}^n \to \mathcal{K}$, defined by* (49).

Proof. If C is convex and closed, by Theorem 3.14, for all $y \in \mathcal{K}^n \setminus C$, we can find an hyperplane (28) containing C and not y. By Remark (3.4), we can chose w', w'', d', d'' in (28) so that $w' \geq w''$ and $d' \geq d''$. Since $\langle w', x\rangle \oplus d' = \langle w'', x\rangle \oplus d''$, for all $x \in C$, $u(x) = (\langle w', x\rangle \oplus d') \ominus (\langle w'', x\rangle \oplus d'') = \mathbb{0}$ for all $x \in C$, so that $\sup_{x\in C} u(x) = \mathbb{0}$. Since $\langle w', y\rangle \oplus d' \neq \langle w'', y\rangle \oplus d''$, and $\langle w', y\rangle \oplus d' \geq \langle w'', y\rangle \oplus d''$ because $w' \geq w''$ and $d' \geq d''$, we must have $\langle w', y\rangle \oplus d' \not\leq \langle w'', y\rangle \oplus d''$. Then, $u(y) = (\langle w', y\rangle \oplus d') \ominus (\langle w'', y\rangle \oplus d'') \not\leq \mathbb{0}$, which shows that C is U-convex.

Conversely, if C is U-convex, Proposition 4.2 shows that C is an intersection of (lower) level sets of differences of affine functions. By Proposition 4.4, these (lower) level sets all are affine hyperplanes, and a fortiori, are convex sets. Moreover, by Proposition 3.7, affine hyperplanes are closed, so C is closed and convex. □

Theorem 4.8. *A function $f : \mathcal{K}^n \to \bar{\mathcal{K}}$ is convex and lower semi-continuous if, and only if, it is a sup of differences of affine functions, i.e., a U-convex function, where U is the set of functions of the form* (49).

The proof relies on the following extension to the case of functions with values in a partially ordered set, of a well known characterization of abstract convexity of functions, in terms of "separation" (see [DK78, Prop. 1.6i], or [Sin97, Th. 3.1, Eqn (3.31)]).

Lemma 4.9. *For any set X, and $U \subset \mathcal{K}^X$, a map $f : X \to \bar{\mathcal{K}}$ is U-convex if, and only if, for each $(x, \nu) \in X \times \mathcal{K}$ such that $f(x) \not\leq \nu$, there exists $u \in U$ such that*

$$u \leq f, \qquad u(x) \not\leq \nu \ . \tag{51}$$

Proof. Let $g = \sup_{u\in U,\, u\leq f} u$. Note first that f is U-convex if, and only if, for all $x \in X$, $g(x) = f(x)$, or equivalently, $g(x) \geq f(x)$ (the other inequality always holds). Recall that for all $t \in \bar{\mathcal{K}}$, $\mathrm{Up}(t) = \{s \in \bar{\mathcal{K}} \mid s \geq t\}$ denotes the upper set generated by t. Trivially: $s \geq t \Leftrightarrow \mathrm{Up}(s) \subset \mathrm{Up}(t) \Leftrightarrow \bar{\mathcal{K}} \setminus \mathrm{Up}(s) \supset \bar{\mathcal{K}} \setminus \mathrm{Up}(t)$. Applying this to $s = g(x)$ and $t = f(x)$, we rewrite $g(x) \geq f(x)$ as

$$f(x) \not\leq \nu \implies g(x) = \sup_{u\in U,\, u\leq f} u(x) \not\leq \nu \ . \tag{52}$$

Since $\sup_{u\in U,\, u\leq f} u(x) \not\leq \nu$ if, and only if, $u(x) \not\leq \nu$ for some $u \in U$ such that $u \leq f$, and since it is enough to check the implication (52) when $\nu \in \mathcal{K}$ (if ν is the top element of $\bar{\mathcal{K}}$, the implication (52) trivially holds), the lemma is proved. □

Proof of Theorem 4.8. $2 \Rightarrow 1$. By Proposition 4.6 and Proposition 4.1, every sup of functions belonging to U is convex and lower semi-continuous.

$1 \Rightarrow 2$. Assume that $f : \mathcal{K}^n \to \bar{\mathcal{K}}$ is convex and lower semi-continuous, and let us prove that f is U-convex.

As mentioned above, the epigraph of f, $\operatorname{epi} f$, is a convex closed subset of $\mathcal{K}^n \times \mathcal{K}$. Consider $(y, \nu) \in \mathcal{K}^n \times \mathcal{K} \setminus \operatorname{epi} f$, so that $f(y) \not\leq \nu$. By Theorem 3.14, there exist (w', μ', d') and (w'', μ'', d'') in $\mathcal{K}^n \times \mathcal{K} \times \mathcal{K}$ with $(w', \mu', d') \geq (w'', \mu'', d'')$ such that

$$\langle (w', \mu'), (z, \lambda)\rangle \oplus d' \leq \langle (w'', \mu''), (z, \lambda)\rangle \oplus d'' \ , \quad \forall (z, \lambda) \in \operatorname{epi} f \ ,$$

$$\langle (w', \mu'), (y, \nu)\rangle \oplus d' \not\leq \langle (w'', \mu'), (y, \nu)\rangle \oplus d'' \ ,$$

that is,

$$\langle w', z\rangle \oplus \mu'\lambda \oplus d' \leq \langle w'', z\rangle \oplus \mu''\lambda \oplus d'' \;, \quad \forall (z,\lambda) \in \operatorname{epi} f \;, \tag{53}$$

$$\langle w', y\rangle \oplus \mu'\nu \oplus d' \not\leq \langle w'', y\rangle \oplus \mu''\nu \oplus d'' \;. \tag{54}$$

Since the function identically equal to the top element of $\bar{\mathcal{K}}$ is trivially U-convex, we shall assume that $f \not\equiv \top\bar{\mathcal{K}}$, i.e., $\operatorname{epi} f \neq \emptyset$. Then, we claim that

$$\mu' = \mu'' \;. \tag{55}$$

Indeed, since $\mu' \geq \mu''$, we may assume that $\mu' \neq \mathbb{0}$. Then, taking $(z,\lambda) \in \operatorname{epi} f$, with λ so large that $\langle z, w'\rangle \oplus \lambda\mu' \oplus d' = \lambda\mu'$, from (53) we obtain $\lambda\mu' \leq \langle z, w''\rangle \oplus \lambda\mu'' \oplus d''$. Then, we cannot have $\mu'' = \mathbb{0}$ (otherwise, $\lambda\mu'$ would be bounded above independently of λ). Therefore, for λ large enough $\langle z, w''\rangle \oplus \lambda\mu'' \oplus d'' = \lambda\mu''$, hence, $\lambda\mu' \leq \lambda\mu''$, which, by $\mu' \geq \mu''$, implies $\lambda\mu' = \lambda\mu''$, and multiplying by λ^{-1}, we get (55).

Hence, by (53)–(55), we have

$$\langle w', z\rangle \oplus \mu'\lambda \oplus d' \leq \langle w'', z\rangle \oplus \mu'\lambda \oplus d'' \;, \quad \forall (z,\lambda) \in \operatorname{epi} f \;, \tag{56}$$

$$\langle w', y\rangle \oplus \mu'\nu \oplus d' \not\leq \langle w'', y\rangle \oplus \mu'\nu \oplus d'' \;. \tag{57}$$

Let

$$\operatorname{dom} f = \{x \in \mathcal{K}^n \mid f(x) \neq \top\bar{\mathcal{K}}\} = \{x \in \mathcal{K}^n \mid (x,\lambda) \in \operatorname{epi} f \text{ for some } \lambda \in \mathcal{K}\} \;.$$

We claim that if $y \in \operatorname{dom} f$, then $\mu' \neq \mathbb{0}$. Indeed, if $\mu' = \mathbb{0}$, then (56) and (57) become

$$\langle w', z\rangle \oplus d' \leq \langle w'', z\rangle \oplus d'' \;, \quad \forall (z,\lambda) \in \operatorname{epi} f \;, \tag{58}$$

$$\langle w', y\rangle \oplus d' \not\leq \langle w'', y\rangle \oplus d'' \;; \tag{59}$$

but, (58), together with $(w', d') \geq (w'', d'')$, yields $\langle w', z\rangle \oplus d' = \langle w'', z\rangle \oplus d''$, for all $(z,\lambda) \in \operatorname{epi} f$, that is, $\langle w', \cdot\rangle \oplus d' = \langle w'', \cdot\rangle \oplus d''$ when restricted to arguments lying in $\operatorname{dom} f$, in contradiction with (59), provided $y \in \operatorname{dom} f$. This proves the claim $\mu' \neq \mathbb{0}$ in this case.

In (56), (57), we may now assume that $\mu' = \mathbb{1}$. Indeed, multiply by $(\mu')^{-1}$, and rename $(\mu')^{-1}w', (\mu')^{-1}w^{''}, (\mu')^{-1}d', (\mu')^{-1}d''$ as w', w'', d', d'' respectively. Now (56) and (57) read

$$\langle w', z\rangle \oplus \lambda \oplus d' \leq \langle w'', z\rangle \oplus \lambda \oplus d'' \;, \quad \forall (z,\lambda) \in \operatorname{epi} f \;, \tag{60}$$

$$\langle w', y\rangle \oplus \nu \oplus d' \not\leq \langle w'', y\rangle \oplus \nu \oplus d' \;. \tag{61}$$

Equation (60) implies that

$$\langle w', z\rangle \oplus d' \leq \langle w'', z\rangle \oplus \lambda \oplus d'' \;, \quad \forall (z,\lambda) \in \operatorname{epi} f \;,$$

whence, by (11),

$$(\langle w', z\rangle \oplus d') \;{\circ\!\!\!-}\; (\langle w'', z\rangle \oplus d'') \leq \lambda \;, \quad \forall (z,\lambda) \in \operatorname{epi} f \;, \tag{62}$$

and hence, defining

$$u := (\langle w', \cdot\rangle \oplus d') \;{\circ\!\!\!-}\; (\langle w'', \cdot\rangle \oplus d'') \in U \;, \tag{63}$$

and using that $f(z) = \bot\{\lambda \mid (z,\lambda) \in \operatorname{epi} f\}$, from (62) we see that $f(z) \geq u(z)$ for $z \in \operatorname{dom} f$; for $z \notin \operatorname{dom} f$, $f(z) = \top\bar{\mathcal{K}}$ and this inequality is trivial, thus we have obtained the first half of (51).

From (61), we deduce that

$$\langle w', y\rangle \oplus d' \not\leq \langle w'', y\rangle \oplus d'' \oplus \nu$$

(because $a \leq b \oplus \nu \implies a \oplus \nu \leq b \oplus \nu$), whence, by (11) (in fact, its equivalent negative form) and (63), we obtain

$$u(y) = (\langle w', y\rangle \oplus d') \ominus (\langle w'', y\rangle \oplus d'') \not\leq \nu,$$

that is, the second part of (51).

For the proof to be complete, we have to handle the case when $y \notin \operatorname{dom} f$ which implies that $(y, \nu) \notin \operatorname{epi} f$. The previous arguments hold true up to a certain point when we cannot claim that $\mu' \neq \mathbb{0}$. Either $\mu' \neq \mathbb{0}$ indeed, and the proof is completed as previously, or $\mu' = \mathbb{0}$, and then (60)–(61) boil down to

$$\langle w', z\rangle \oplus d' \leq \langle w'', z\rangle \oplus d'' , \quad \forall z \in \operatorname{dom} f , \tag{64}$$

$$\langle w', y\rangle \oplus d' \not\leq \langle w'', z\rangle \oplus d'', \tag{65}$$

without having to redefine the original (w', d', w'', d''). For any $\alpha \in \mathcal{K} \setminus \{\mathbb{0}\}$, define the functions

(66)

$$u_\alpha(\cdot) = \alpha(\langle w', \cdot\rangle \oplus d') \ominus \alpha(\langle w'', \cdot\rangle \oplus d'') = \alpha\big((\langle w', \cdot\rangle \oplus d') \ominus (\langle w'', \cdot\rangle \oplus d'') \big) ,$$

which all belong to U. Because of (64), all those functions are identically equal to $\mathbb{0}$ over $\operatorname{dom} f$, hence they are trivially less than or equal f over $\operatorname{dom} f$ but also over the whole $\mathcal{K}^n$. On the other hand, because of (65),

$$(\langle w', y\rangle \oplus d') \ominus (\langle w'', y\rangle \oplus d'') > \mathbb{0} .$$

Multiplying this strict inequality by a large enough α, and using (66), we see that given ν, there exists α for which $u_\alpha(y) > \nu$, and a fortiori, $u_\alpha(y) \not\leq \nu$. The proof is now complete. □

Remark 4.10. By Remark 3.4, Theorem 4.8 remains valid if the set U is be replaced by the subset of the functions in (49) such that $w' \geq w''$ and $d' \geq d''$.

An illustration of Theorem 4.8 has been given in Figure 1 in the introduction: the figure shows a convex function over $\mathbb{R}_{\max}$, together with its supporting hyperplanes (which are epigraphs of differences of affine functions, whose shapes were already shown in Figure 2).

References

[AS03] M. Akian and I. Singer. Topologies on lattice ordered groups, separation from closed downward sets, and conjugations of type lau. 2003. Submitted.

[BCOQ92] F. Baccelli, G. Cohen, G. Olsder, and J. Quadrat. *Synchronization and Linearity — an Algebra for Discrete Event Systems*. Wiley, 1992.

[Bir67] G. Birkhoff. *Lattice Theory*, volume XXV of *American Mathematical Society Colloquium Publications*. A.M.S, Providence, Rhode Island, 1967. (third edition).

[BJ72] T. Blyth and M. Janowitz. *Residuation Theory*. Pergamon press, 1972.

[CG79] R. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer, 1979.

[CGQ96] G. Cohen, S. Gaubert, and J. Quadrat. Kernels, images and projections in dioids. In *Proceedings of WODES'96*, Edinburgh, August 1996. IEE.

[CGQ97] G. Cohen, S. Gaubert, and J. Quadrat. Linear projectors in the max-plus algebra. In *5th IEEE Mediterranean Conference on Control and Systems*, Paphos, Cyprus, 1997.

[CGQ99] G. Cohen, S. Gaubert, and J. Quadrat. Max-plus algebra and system theory: where we are and where to go now. *Annual Reviews in Control*, 23:207–219, 1999.

[CGQ01] G. Cohen, S. Gaubert, and J. Quadrat. Duality of idempotent semimodules. In *Proceedings of the Workshop on Max-Plus Algebras, IFAC SSSC'01*, Praha, 2001. Elsevier.

[CGQ02] G. Cohen, S. Gaubert, and J. Quadrat. Duality and separation theorem in idempotent semimodules. Research Report 4668, INRIA, December 2002. Also e-print arxiv:math.FA/0212294. To appear in Linear Algebra and Appl.

[CKR84] Z. Cao, K. Kim, and F. Roush. *Incline algebra and applications*. Ellis Horwood, 1984.

[DJLC53] M. Dubreil-Jacotin, L. Lesieur, and R. Croisot. *Leçons sur la Théorie des Treillis, des Structures Algébriques Ordonnées, et des Treillis géométriques*, volume XXI of *Cahiers Scientifiques*. Gauthier Villars, Paris, 1953.

[DK78] S. Dolecki and S. Kurcyusz. On Φ-convexity in extremal problems. *SIAM J. Control Optim.*, 16:277–300, 1978.

[Fan63] K. Fan. On the Krein Milman theorem. In V. Klee, editor, *Convexity*, volume 7 of *Proceedings of Symposia in Pure Mathematics*, pages 211–220, AMS, Providence, 1963.

[GHK+80] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott. *A Compendium of Continuous Lattices*. Springer, 1980.

[GM02] M. Gondran and M. Minoux. *Graphes, Dioïdes et semi-anneaux*. TEC & DOC, Paris, 2002.

[Gol92] J. Golan. *The theory of semirings with applications in mathematics and theoretical computer science*, volume 54. Longman Sci & Tech., 1992.

[Gun03] J. Gunawardena. From max-plus algebra to nonexpansive mappings: a nonlinear theory for discrete event systems. *Theor. Comput. Sci.*, 293:141–167, 2003.

[KM97] V. N. Kolokoltsov and V. P. Maslov. *Idempotent analysis and applications*. Kluwer Acad. Publisher, 1997.

[Kor65] A. A. Korbut. Extremal spaces. *Dokl. Akad. Nauk SSSR*, 164:1229–1231, 1965.

[LMS01] G. Litvinov, V. Maslov, and G. Shpiz. Idempotent functional analysis: an algebraical approach. *Math. Notes*, 69(5):696–729, 2001. Also eprint arXiv:math.FA/0009128.

[LS02] G. Litvinov and G. Shpiz. Nuclear semimodules and kernel theorems in idempotent analysis: an algebraic approach. *Doklady Math. Sci.*,, 6, 2002. Also math.FA/0206026.

[Mas73] V. P. Maslov. *Méthodes Opératorielles*. Mir, Moscou, 1973. trad. fr. 1987.

[MLRS02] J.-E. Martínez-Legaz, A. M. Rubinov, and I. Singer. Downward sets and their separation and approximation properties. *J. Global Optim.*, 23(2):111–137, 2002.

[MLS91] J.-E. Martínez-Legaz and I. Singer. ∨-dualities and ⊥-dualities. *Optimization*, 22:483–511, 1991.

[MN91] P. Meyer-Nieberg. *Banach lattices*. Springer, 1991.

[Mol88] P. Moller. *Théorie algébrique des Systèmes à Événements Discrets*. Thèse, École des Mines de Paris, 1988.

[MS92] V. P. Maslov and S. N. Samborskiĭ. *Idempotent analysis*, volume 13 of *Advances In Soviet Mathematics*. Amer. Math. Soc., Providence, 1992.

[Rom67] I. V. Romanovskiĭ. Optimization of the stationary control for a discrete deterministic process. *Kibernetika*, 2:66–78, 1967.

[RS00] A. M. Rubinov and I. Singer. Topical and sub-topical functions, downward sets and abstract convexity. *Optimization*, 50:307–351, 2000.

[Rub00] A. M. Rubinov. *Abstract convexity and global optimization*. Kluwer, 2000.

[Sin84] I. Singer. Generalized convexity, functional hulls and applications to conjugate duality in optimization. In G. Hammer and D. Pallaschke, editors, *Selected Topics in Operations Research and Mathematical Economics*, number 226 in Lecture Notes Econ. Math. Systems, pages 49–79. Springer, 1984.

[Sin97] I. Singer. *Abstract convex analysis*. Wiley, 1997.

[SS92] S. N. Samborskiĭ and G. B. Shpiz. Convex sets in the semimodule of bounded functions. In *Idempotent analysis*, pages 135–137. Amer. Math. Soc., Providence, RI, 1992.

[Vor67] N. N. Vorobyev. Extremal algebra of positive matrices. *Elektron. Informationsverarbeit. Kybernetik*, 3:39–71, 1967.

[Vor70] N. N. Vorobyev. Extremal algebra of non-negative matrices. *Elektron. Informationsverarbeit. Kybernetik*, 6:303–311, 1970.

[Wag91] E. Wagneur. Moduloids and pseudomodules. 1. dimension theory. *Discrete Math.*, 98:57–73, 1991.

[Zim76] K. Zimmermann. *Extremální Algebra.* Ekonomický ùstav ČSAV, Praha, 1976. (in Czech).

[Zim77] K. Zimmermann. A general separation theorem in extremal algebras. *Ekonom.-Mat. Obzor*, 13(2):179–201, 1977.

[Zim79a] K. Zimmermann. Extremally convex functions. *Wiss. Z. Päd. Hochschule "N. K. Krupskaya"*, 17:3–7, 1979.

[Zim79b] K. Zimmermann. A generalization of convex functions. *Ekonom.-Mat. Obzor*, 15(2):147–158, 1979.

[Zim81] U. Zimmermann. *Linear and Combinatorial Optimization in Ordered Algebraic Structures.* North Holland, 1981.

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