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Max-Plus-Times Linear Systems

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1 Description of the problem

Let A and B be (m, n) matrices with real nonnegative entries. Let C and D be (p, n) matrices with entries in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. We denote by $\otimes$ the max-plus matrix product defined by

$$[E \otimes F]_{ij} = \max_k (E_{ik} + F_{kj}) \ .$$

Let δ denote the backward shift operator on sequences $x = (x_k)_{k\in\mathbb{Z}}$ with entries in $\overline{\mathbb{R}}$, defined by $(\delta x)_k = x_{k-1}$. Let $A(\delta)$,$B(\delta)$, [resp. $C(\delta)$ and $D(\delta)$] be matrices whose entries are monomials [resp. max-plus monomials] in δ with nonnegative coefficients [resp. with coefficients in $\overline{\mathbb{R}}$].

We are interested in solving the following problems.

1. Describe the set of n-vectors X with entries in $\overline{\mathbb{R}}$ satisfying

$$\text{(I)} \quad \begin{cases} AX = BX \ , \\ C \otimes X = D \otimes X \ . \end{cases}$$

   In the first equation we adopt the convention $0 \times (-\infty) = 0$.

2. Describe the set of n-vectors of sequences X satisfying

$$\text{(II)} \quad \begin{cases} A(\delta)X = B(\delta)X \ , \\ C(\delta) \otimes X = D(\delta) \otimes X \ . \end{cases}$$

3. Describe the set of couples (λ, X), where X is an n-vector with entries in $\mathbb{R}$ and $\lambda \in \mathbb{R}$, satisfying

$$\text{(III)} \quad \begin{cases} A(\lambda)X = B(\lambda)X \ , \\ C(\lambda) \otimes X = D(\lambda) \otimes X \ , \end{cases}$$

   where $A_{ij}(\lambda)$, $B_{ij}(\lambda)$ denote the standard evaluations of the corresponding monomials, and $C_{ij}(\lambda)$, $D_{ij}(\lambda)$ denote the max-plus evaluations of the corresponding monomials (the evaluation of a max-plus monomial $m(\delta) = a\delta^n$ at λ (a real number) is defined by $m(\lambda) = n\lambda + a$).

2 Motivations

Such problems arise in at least two different contexts.

1. *Markov Decision processes.* Classical stochastic dynamic programming equations correspond to the second problem (II). Indeed we can partition the vector X into (Y, Z). Then, choosing the matrices $A(\delta) = (I, 0)$, $B(\delta) = (0, B')$, $C(\delta) = (\epsilon, E)$, $D(\delta) = (\delta D', \epsilon)$ (where I is the standard identity matrix, E the max-plus identity matrix and ϵ the zero max-plus matrix), System (II) describes the recurrence
$$\begin{cases} Y_k = B' Z_k \ , \\ Z_k = D' \otimes Y_{k-1} \ . \end{cases}$$
If we are interested in the component Z we obtain
$$Z_k = D' \otimes (B' Z_{k-1}) \ ,$$
which is a standard stochastic dynamic programming equation as soon as $B'1 = 1$. The asymptotics of these problems when n goes to ∞ leads to Problem (III). Indeed, the equation
$$Z = D' \otimes (B' Z) + \lambda \ ,$$
is a standard stochastic dynamic programming equation for computing the maximal cost by unit of time in the ergodic case [18].

2. *Simulation of general Petri nets*. The dynamic of a general Petri net can be described by special classes of the second type of equations (see [14] Th.II.2), which are more general than the stochastic dynamic programming equations. For some particular routing policies, simulating Petri nets is equivalent to solving stochastic dynamic programming equations (see [4]).

3 Available results

Clearly a lot of results are known in particular cases, but the general theory does not exist.

1. When C and D are max-plus zero matrices, we are in the standard linear algebraic situation.

2. When A and B are conventional zero matrices, we are in the max-plus linear situation.

   (a) When C is the max-plus identity matrix, Problem (II) corresponds to deterministic dynamic programming.

   (b) When the matrix D has only one max-plus nonzero column, Problem (I) can be solved using residuation theory (see for example [2], [1, Ch.4.]).

   (c) A Cramer theory exists for Problem (I) with general C and D matrices (see [1, Ch.3 Sect.4],[10, Ch.3],[15]). This problem can also be solved by elimination methods [3, 11],[10, Ch.3].

   The references [5, 6] may be useful to understand the kernels and the images of max-plus linear operators. See also [17, 12] for available results on semimodules and semirings.

3. Some special instances of Problem (I) are seen in [9, Ch.3 and Ch.4] as extended linear complementary problems. The set of solutions, which is an union of faces of polyedra, cannot be simple in full generality. A kind of max-plus algebraic geometry has to be developped for solving this problem for matrices with integer entries. Some preliminary results on max-plus polynomials can be found in [1, Ch.3 Sect.6],[8, Sec. VIII].

4. Pure standard algebra or max-plus eigenvalues problems are understood, see [7, 13, 16, 10, 1] for the max-plus case. The Markov decision process case is also standard [18]. The problem with simultaneous dependence, in δ, of A in one hand, and C and D in the other hand, is not homogeneous and may have no practical interest. For example, in the stochastic dynamic programming case, B and A do not depend of δ.

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