Soumis à IEEE-AC 96

TIMED EVENT GRAPHS WITH MULTIPLIERS AND HOMOGENEOUS MIN-PLUS SYSTEMS

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ABSTRACT. We study fluid analogues of a subclass of Petri nets, called Fluid Timed Event Graphs with Multipliers, which are a timed extension of weighted T-Systems studied in the Petri Net literature. These event graphs can be studied naturally, with a new algebra, analogous to the min-plus algebra, but defined on piecewise linear concave increasing functions, endowed with the pointwise minimum as addition, and the composition of functions as multiplication. A subclass of dynamical systems in this algebra, which have a property of homogeneity, can be reduced to standard min-plus linear systems after a change of counting units. We give a necessary and sufficient condition under which a fluid timed event graph with multipliers can be reduced to a fluid timed event graph without multipliers. In the fluid case, this class corresponds to the so-called expansible timed event graphs with multipliers of A. Munier, or to conservative weigthed T-systems. The change of variable is called here a potential. Its restriction to the transitions nodes of the event graph is a T-semiflow.

Key words. Timed Petri Nets, Timed Event Graphs, Dynamic Programming, Discrete Event Systems, Max-Plus Algebra, Potentials, Weighted T-Systems.

1. INTRODUCTION

An event graph is a Petri net such that each place has only one input arc and one output arc. If the tokens have to stay a minimum amount of time in the places, we speak of *Timed* Event Graph (TEG). These TEGs are well adapted for modeling synchronizations. In many systems, synchronization is essential. In manufacturing, in order to start a task, a machine and a part must be both ready. In computer science, in order to achieve a computation, we need a processor and an information.

Several units of the same resource may be required to achieve a specific task. Then, the corresponding event graph consumes or produces more than one token in adjacent places, at each transition firing. The corresponding event graph is called a Timed Event Graph with Multipliers (TEGM). To assemble a bicycle, two wheels, a frame and a certain amount of manpower are needed. In a chemical process, a reaction producing a molecule consumes in general more than one atom of a given sort.

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Synchronization is not specific to discrete systems, and we will consider here Fluid analogues of Timed Event Graphs with Multipliers (FTEGM) in which fluids circulate instead of tokens. For instance, in chemical processes, synchronization (stoichiometry here) is essential and the products used in a chemical reaction may be fluids.

We give some mathematical tools well suited to manipulate FTEGM. In particular, very briefly, we introduce a new kind of power series, extending that considered in [1], which allow us to express the input-output relations of FETGM (in [7], a systematic classification of all the kinds of power series that may pop up in Petri net modeling is presented). These power series are elements of a new noncommutative min-plus algebra: the set of piecewise linear concave functions, endowed with the pointwise minimum as addition, and the composition of functions as multiplication. This is the mathematical cost to pay for dealing with multipliers. This is a somewhat expensive cost and it is natural to try to find particular cases for which this can be avoided.

In [14], A. Munier introduced and studied an important subclass of TEGM, that we will call conservative, in which the product of multipliers along any circuit is equal to one. The main result of [14] reduces such a TEGM to a conventional TEG after a duplication of transitions. In the context of Petri nets where only the logical aspect is considered, this subclass is known as conservative weihted T-systems [17, 13].

When the multipliers derive from a potential (a vector indexed by both places and transitions), the dynamic of a FTEGM can be reduced to classical min-plus linear recurrent equations by a diagonal change of variables, given by the potential. The existence of a potential is equivalent to the property pointed out by A. Munier. The restriction of a potential to transitions is called a semiflow in the Petri net literature [17]. In the example of the bicycle, counting pairs of wheels instead of wheels is quite natural. It is important to remark that this change of variables, called linearization, is a min-plus algebra nonlinear operation. However with this way of counting the dynamic becomes linear.

As a by-product of the linearization, the existence of an eventual periodic regime is readily obtained, the performance being characterized in terms of invariants of the original net. We also show that linearizable FTEGM are characterized by an input-output homogeneity property, which is essentially a conservation law between input and output quantities.

The fluid case, considered here, is much simpler than the discrete case considered by Munier. The linearization procedure does not increase the number of transitions of the system, while the expansion procedure of [14] results in a blow up of the number of transitions.

The fact that a fluid approximation is considered in the case of discrete systems may have an impact on the liveness of the Petri net. For instance, if a single wheel is available, in reality, the production is blocked, but the fluid model gives a production of one half of bicycle. This liveness issue is solved by Munier [14] (see also [4] for nets with a single circuit). In this paper, we show that the discrete and fluid behaviors coincide when the integer marking is a multiple of a certain minimal marking. Finding good units for counting has nothing to do with liveness but with flows. Fluid analogues are suited to find these units but not to study liveness problems.

The paper is organized as follows. In section 2, we recall classical definitions about Petri nets and introduce the subclass of TEGs with Multipliers. In section 3, we introduce an algebra of operators which yields a simple representation of FTEGM. In section 4, we state the main results of the paper, namely, the characterization of linearizable FTEGM, periodic regime, performance (periodic thoughput), invariants. In Appendix 6.1, we state and prove an elementary lemma about potentials on graphs which is the algebraic core of the properties presented here. In Appendix 6.2, the main results are proved as mere consequences of this lemma. Some open ends are pointed out in conclusion.

2. RECURRENT EQUATIONS OF TIMED EVENT GRAPHS WITH MULTIPLIERS

We begin by recalling the usual definition of Timed Petri Nets and Event Graphs.

DEFINITION 2.1 (TPNM,TEGM,TEG). A Timed Petri Net with Multipliers (TPNM) is a valued bipartite graph given by a 5-tuple $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, M, m, \tau)$.

1. The finite set $\mathcal{P}$ is called the set of places. *A place may contain tokens which travel from place to place according to a firing process described later on.*
2. The finite set $\mathcal{Q}$ is called the set of transitions. *When firing, a transition consumes tokens of the upstream places and produces tokens in the downstream places.*
3. The set of nodes is $\mathcal{R} \stackrel{\text{def}}{=} \mathcal{P} \cup \mathcal{Q}$.
4. The matrix $M \in \mathbb{N}^{\mathcal{R}\times\mathcal{R}}$ is called incidence matrix. *The integer M_{pq} (resp. M_{qp}) denotes the number of edges from transition q to place p (resp. from place p to transition q).* Since the graph is bipartite the blocks $M_{\mathcal{P}\mathcal{P}}$ and $M_{\mathcal{Q}\mathcal{Q}}$ are zero. We denote by r^{out} the set of vertices (places or transitions) downstream a vertex r and r^{in} the set of vertices upstream r. Formally, $r^{\text{out}} = \{s \mid M_{sr} \neq 0\}$, $r^{\text{in}} = \{s \mid M_{rs} \neq 0\}$.
5. The vector $m \in \mathbb{N}^{\mathcal{P}}$ is called initial marking. *The integer m_p denotes the number of tokens being initially in place p .*
6. The vector $\tau \in \mathbb{N}^{\mathcal{P}}$ is called holding time. *The integer τ_p gives the minimal time a token must spend in place p before becoming available for consumption by downstream transitions*[1].

[1]Without loss of modeling power, the firing of transitions is supposed to be instantaneous (i.e. it involves no delay in consuming and producing tokens).

7. A Timed Event Graph with Multipliers (TEGM) is a TPNM such that there is exactly one transition upstream and one transition downstream each place.
8. An (ordinary) Timed Event Graph (TEG) is a TEGM with unit multipliers (i.e. $M_{rs} \in \{0, 1\}$).

The (earliest) functioning of a TEGM is as follows. A transition q *fires* as soon as all the places p upstream q contain enough tokens (M_{qp}) having spent at least τ_p units of time in place p (earliest firing rule). When the transition q fires, it consumes M_{qp} tokens in each upstream place p and produces $M_{p'q}$ tokens in each downstream place p'.

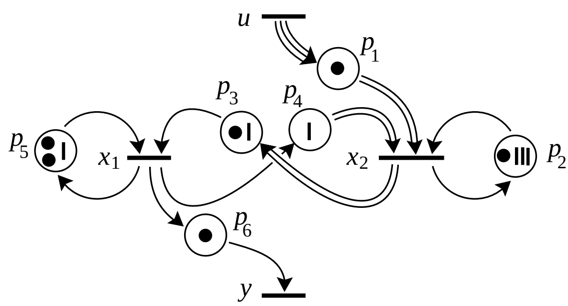

FIGURE 1. A timed event graph with multipliers.

DEFINITION 2.2. 1. With each place p (resp. transition q), a *counter variable* Z_p (resp. Z_q) is associated. It denotes the cumulated number of tokens which have entered place p (resp. number of firings of transition q) up to time[2] $t \in \mathbb{Z}$.
2. We call μ the *multiplier matrix* , i.e. the $\mathcal{R} \times \mathcal{R}$ matrix with values in $\mathbb{R}^+$ and entries $\mu_{pq} \stackrel{\text{def}}{=} M_{pq}$, $\mu_{qp} \stackrel{\text{def}}{=} M_{qp}^{-1}$ for $p \in \mathcal{P}, q \in \mathcal{Q}$, $M_{qp} \neq 0$, with $\mu_{rr'} = 0$ if $M_{rr'} = 0, r, r' \in \mathcal{R}$.

ASSERTION 2.3. *The counter variables of a TEGM*[3] *(under the earliest firing rule) satisfy the following equations:*

$$Z_q(t) = \min_{p \in q^{\text{in}}} \lfloor \mu_{qp}(Z_p(t - \tau_p) + m_p) \rfloor, \quad Z_p(t) = \mu_{pp^{\text{in}}} Z_{p^{\text{in}}}(t) \ , \tag{2.1}$$

with $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} \mid \ n \leq x\}$.

Eliminating Z_p, we get the *transition-to-transition* equations:

$$Z_q(t) = \min_{p \in q^{\text{in}}} \lfloor \nu_p + \alpha_p Z_{p^{\text{in}}}(t - \tau_p) \rfloor \ , \tag{2.2}$$

with the notation $\nu_p = \mu_{p^{\text{out}}p} m_p$, $\alpha_p = \mu_{p^{\text{out}}p} \mu_{pp^{\text{in}}}$. Dually, place-to-place equations can be obtained.

[2]We consider here only $t \in \mathbb{Z}$ but the result could be generalized to $t \in \mathbb{R}$.

[3]We shall only give here the dynamic equations of TEGM. Indeed, the general Petri net equations exhibit a higher order of complexity due to the presence of routing decisions; see [7, 2].

The behavior of this dynamic is extremely simple in the TEG case. Then, the multipliers are equal to one, and (2.1) becomes:

$$Z_q(t) = \min_{p \in q^{\text{in}}} \left(Z_p(t - \tau_p) + m_p \right) , \quad Z_p(t) = Z_{p^{\text{in}}}(t) \ . \tag{2.3}$$

Note that the integer part has been dropped, since the dynamics (2.3) obviously preserves integrity. Therefore, fluid TEG need not be distinguished from conventional TEG. The behavior of TEGs is well understood [6, 5, 1]. In particular, strongly connected TEG reach a periodic regime after a finite time, and the corresponding periodic throughput can be easily determined. This leads us to raise the following question, to which the remaining part of the paper is devoted: *when does a TEGM reduce to a TEG by a change of variables ?*

We shall only consider here *diagonal* changes of variables (change of units): $Z = DZ'$ (where D is a diagonal matrix with positive diagonal entries). This restriction can be justified theoretically. We need a change of variables which preserves the min-plus structure of the equations. Thus, we need a matrix P such that the changes of variables $Z = PZ'$ and $Z' = P^{-1}Z$ are *order preserving*, which is possible if and only if P is a monomial matrix (product of a diagonal matrix with positive entries and a permutation matrix). Indeed the inverse of a nonnegative matrix is nonnegative if and only if it is a monomial matrix (see [3], Lemma 4.3, p.68).

REMARK 2.4. If $\tau_p = 0$ for some places, (2.1) becomes an implicit system and we may have difficulties in proving the existence of a finite solution. For the subclass of systems (with the potential property) discussed here, the existence will follow at once from the existing results about TEG. In general, it is not too restrictive to assume that : *there are no circuits containing only places with zero holding times.* We will call such graphs *explicit.* For explicit FTEGM, (2.1) has a unique solution, which can be shown by adapting the argument of [1], Lemma 2.65, p.78.

DEFINITION 2.5 (INPUT-OUTPUT PARTITION). We partition the set of transitions $\mathcal{Q} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{U}$ is the set of transitions with no predecessors (input transitions), $\mathcal{Y}$ is the set of transitions with no successors (output transitions) and $\mathcal{X} = \mathcal{Q} \setminus (\mathcal{U} \cup \mathcal{Y})$ (state transitions). We denote by u (resp. x, y) the vector of input (resp. state, output) counters $Z_q, q \in \mathcal{U}$ (resp. $\mathcal{X}$, $\mathcal{Y}$).

Throughout the paper, we will study the *input-output* behavior of the system. That is, we will look for the trajectory $(x(\cdot), y(\cdot))$ corresponding to the earliest firing rule (which yields the largest possible values of counters), given an input history $u(\cdot)$. This encompasses the *autonomous regime* traditionally considered in the Petri net literature, when the system is frozen at an initial condition $Z_q(t) = w_q \in \mathbb{R}$ for $t < 0$ (usually $w_q = 0$), and then evolves freely according to the dynamics (2.1) for $t \geq 0$. This can be obtained as a specialization of the input-output case by adjoining an input transition q' (with an associated empty place) upstream each original transition q, setting $u_{q'}(t) = w_q$ for $t < 0$, $u_{q'}(t) = +\infty$ otherwise.

EXAMPLE 2.6. Let us consider the TEGM depicted in Fig. 1. In each place the initial marking is given by the number of tokens (dots), the timing by the number of bars (time units), and the multipliers by the numbers of arcs in parallel. The equations read

$$x_1(t) = \min \lfloor 2 + x_1(t-1), 1 + 2x_2(t-1) \rfloor \,, \quad y(t) = \lfloor 1 + x_1(t) \rfloor \ .$$

$$x_2(t) = \min \lfloor (1/2)x_1(t-1), 1 + x_2(t-3), 1/2 + (3/2)u(t) \rfloor \ ,$$

3. OPERATORIAL REPRESENTATION OF FTEGM

Fluid Timed Event Graphs with Multipliers (FTEGM) are defined as in Def. 2.1, but the marking m and the multipliers M take real values, that is $m_p \in \mathbb{R} \cup \{+\infty\}$, $M_{xy} \in \mathbb{R}^+$. Counter functions are subsequently real valued, and the integer parts vanish, that is

$$Z_q(t) = \min_{p \in q^{\mathrm{in}}} \left(\nu_p + \alpha_p Z_{p^{\mathrm{in}}}(t - \tau_p) \right) \ . \tag{3.1}$$

We next introduce the algebraic tools needed to easily handle such dynamics.

1. A *semiring* is a set, equipped with an addition $\oplus$, and a product $\otimes$, such that: $\oplus$ is associative, commutative, has a zero (denoted ε); $\otimes$ is associative, has a unit (denoted e); $\otimes$ distributes over $\oplus$; zero is absorbing ($\varepsilon \otimes a = \varepsilon$). An idempotent semiring (such that $a \oplus a = a$) is called a *dioid*. A *semifield* is a semiring whose nonzero elements are invertible. E.g. $(\mathbb{R}^+, +, \times)$ and $(\mathbb{R}^+ \cup \{+\infty\}, \max, \times)$ are idempotent semifields.
2. The idempotent semifield $\mathbb{R}_{\min}$ is the set $\mathbb{R} \cup \{+\infty\}$, equipped with min as addition and the usual sum as multiplication, i.e. $a \oplus b = \min(a, b)$, $a \otimes b = a + b$ with $\varepsilon = +\infty$ and $e = 0$.
3. A *signal* is a map $u : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$. We denote by $\mathcal{S}$ the set of signals, which has a structure of min-plus semimodule (the analogue of a module but with scalars belonging to a semiring, here $\mathbb{R}_{\min}$). Counter functions are instances of nondecreasing signals.
4. An *operator* is a map $\mathcal{H} : \mathcal{S} \to \mathcal{S}$. It is *linear* if it preserves the min-plus semimodule structure of signals, that is,

$$\mathcal{H}(u \oplus v) = \mathcal{H}(u) \oplus \mathcal{H}(v) \ , \tag{3.2a}$$
$$\mathcal{H}(\lambda \otimes u) = \lambda \otimes \mathcal{H}(u) \ , \tag{3.2b}$$

for all signals u, v and constants λ (we denote by $\lambda \otimes u$ the signal $t \mapsto \lambda + u(t)$). An operator is *additive* if it only satisfies the relation

(3.2a). The following three families of operators are central in modeling FTEGM.

$$\gamma^\nu : \quad \gamma^\nu x(t) \stackrel{\text{def}}{=} x(t) + \nu \quad \text{(shift in counting)} \tag{3.3a}$$

$$\delta^\tau : \quad \delta^\tau x(t) \stackrel{\text{def}}{=} x(t-\tau) \quad \text{(shift in dating)} \tag{3.3b}$$

$$\mu : \quad \mu x(t) \stackrel{\text{def}}{=} \mu \times x(t) \quad \text{(scaling),} \tag{3.3c}$$

where $\nu \in \mathbb{R} \cup \{+\infty\}$, $\tau \in \mathbb{N}$, $\mu \in \mathbb{R}^+$. We note that γ and δ are linear, while the operators $\mu \neq 1$ are only additive. They satisfy

$$\gamma^\nu \delta^\tau = \delta^\tau \gamma^\nu \ , \quad \mu\delta^\tau = \delta^\tau \mu \ , \quad \mu\gamma^\nu = \gamma^{\mu\nu}\mu \ , \tag{3.4}$$

$$\gamma^\nu \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')} \ ,$$

$$\gamma^\nu \gamma^{\nu'} = \gamma^{\nu+\nu'}, \ \delta^\tau \delta^{\tau'} = \delta^{\tau+\tau'}, \ \mu\mu' = \mu \times \mu' \ .$$

5. We denote by $\mathcal{A}_{\min}$ the (noncommutative) dioid of finite sums of operators $\mu\gamma^n$ (equipped with pointwise min and composition). Thus, an element of $\mathcal{A}_{\min}$ is a map $p = \bigoplus_{i=1}^k \gamma^{\nu_i}\mu_i$, $p(x) = \min_{1\le i\le k}(\nu_i + \mu_i x)$, with $\varepsilon = \gamma^{+\infty}$ and $e = \gamma^0$.
6. We denote by $\mathcal{A}_{\min}[[\delta]]$ (resp. $\mathcal{A}_{\min}[\delta]$) the dioid of power series (resp. polynomials) with coefficients in $\mathcal{A}_{\min}$, with the zero element $\gamma^{+\infty}$ and the identity element $\gamma^0\delta^0$.
7. The subset of $\mathcal{A}_{\min}[[\delta]]$, where all multipliers appearing in a power series satisfy $\mu = 1$, is a dioid called $\mathbb{R}_{\min}[[\delta]]$. The subdioid of polynomials $\mathbb{R}_{\min}[[\delta]] \cap \mathcal{A}_{\min}[\delta]$ is denoted $\mathbb{R}_{\min}[\delta]$.
8. We extend the min-plus matrix product notation to denote the action of matrices of operators on vectors of signals. Given a matrix A with entries in $\mathcal{A}_{\min}[[\delta]]$ and a vector of signals u (with compatible dimensions), we set $(Au)_i \stackrel{\text{def}}{=} \bigoplus_j A_{ij}(u_j)$. The zero (matrix whose entries are identically ε) and identity element (diagonal matrix with e entries on the diagonal and ε elsewhere) are still denoted ε and e.

We next rephrase the dynamic equations of FTEGM algebraically.

ASSERTION 3.1. *The dynamics of a FTEGM can be written*

$$x = Ax \oplus Bu, \quad y = Cx \oplus Du \ , \tag{3.5}$$

where A, B, C, D *are matrices with entries in* $\mathcal{A}_{\min}[\delta]$. *We say that* (A, B, C, D) *is a* representation *of the system. Moreover, in the TEG case, the entries of* A, B, C, D *belong to* $\mathbb{R}_{\min}[\delta]$.

As an immediate corollary of the representation (3.5), one obtains the following input-output representation result, taken from [7].

ASSERTION 3.2 (TRANSFER REPRESENTATION). *For an* explicit[4] *FTEGM with representation* A, B, C, D*, we have*

$$y = Hu, \quad \text{where } H = D \oplus CA^*B \ , \quad A^* = e \oplus A \oplus A^2 \oplus \cdots \tag{3.6}$$

[4]See Remark 2.4. Without this condition, the infinite sum A^* need not converge in $\mathcal{A}_{\min}[[\delta]]$.

The $\mathcal{Y} \times \mathcal{U}$ matrix H with entries in $\mathcal{A}_{\min}[[\delta]]$ (in $\mathbb{R}_{\min}[[\delta]]$ in the TEG case) is called transfer matrix.

In other words, the input-output behavior of the system is summarized by a matrix of power series. The series obtained as entries of transfer matrices of systems of type (3.5) form the strict subclass of *rational* series. See [7] for more details.

Let us recall the classical characterization of these (rational) transfer series in terms of paths. The weight of a path π, denoted $|\pi|$, is equal to the product of the operators of types γ, δ, μ (associated with arcs), taken along the path. We note that $|\pi| \in \mathcal{A}_{\min}[\delta]$ is indeed a monomial, and we write

$$|\pi| = \gamma^{\nu_\pi} \delta^{\tau_\pi} |\pi|_\mu \quad , \tag{3.7}$$

where $|\pi|_\mu \in \mathbb{R}^+$ is the *multiplicative weight* of the path (i.e. the product of the multipliers along the arcs), $\tau_\pi \in \mathbb{N}$ is the *sum of the holding times* of the places of the path, and the "discounted marking" $\nu_\pi \in \mathbb{R}$ is obtained, from the original marking, by applying several times the commutation rules. The set of paths from i to j is denoted $\wp_{ji}$. The following elementary fact needs no formal proof.

ASSERTION 3.3. *For all input and output transitions $i \in \mathcal{U}$, $j \in \mathcal{Y}$, the transfer series H_{ji} from i to j is equal to the sum of the weights of the paths $\wp_{ji}$ from i to j:*

$$H_{ji} = \bigoplus_{\pi \in \wp_{ji}} |\pi| = \bigoplus_{\pi \in \wp_{ji}} \gamma^{\nu_\pi} \delta^{\tau_\pi} |\pi|_\mu \quad . \tag{3.8}$$

EXAMPLE 3.4. The TEGM depicted in Fig. 1 admits the following representation.

$$A = \begin{pmatrix} \gamma^2\delta & \gamma\delta 2 \\ (1/2)\delta & \gamma\delta^3 \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon \\ (1/2)\gamma 3 \end{pmatrix}, \quad C = \begin{pmatrix} \gamma & \varepsilon \end{pmatrix}, \quad D = \begin{pmatrix} \varepsilon \end{pmatrix}.$$

The transfer will be explicitly computed in Ex. 4.7 below.

REMARK 3.5 (DYNAMIC PROGRAMMING INTERPRETATION). We note that (3.1) can be interpreted as the dynamic programming equation of a deterministic Markov decision process with control dependent discount rate α and cost ν. The transitions of the event graph are the states. The control chooses between the upwards arcs of the transitions. Then, Z is the Bellman function of the corresponding dynamic programming equation. In particular, the results given below characterize the subclass of discounted decision problems which reduce to the undiscounted case after a diagonal change of variables. This dynamic programming interpretation is detailed in [7].

4. LINEARIZABILITY AND EXISTENCE OF POTENTIAL

We next introduce some elementary notions, needed to state the main result.

DEFINITION 4.1. Let A denote an $n \times n$ matrix, with entries in a semifield[5] $\mathcal{K}$.

1. We call *transition graph* of A, the (directed) graph $\mathcal{G}(A)$ with nodes $\{1, \dots, n\}$, and arcs $j \mapsto i$ whenever $A_{ij} \neq \varepsilon$. A path from j to i in the transition graph $\mathcal{G}(A)$ is denoted π_{ij}. Its *weight* is denoted $|\pi_{ij}|_A \stackrel{\text{def}}{=} A_{il_n} A_{l_n l_{n-1}} \cdots A_{l_1 j}$, if $\pi_{ij} = (j, l_1, l_2, \dots, l_n, i)$.
2. The matrix A is *conservative* if the multiplicative weight of a path depends only on the initial and final nodes of the path i.e. if for all paths π_{ij} from j to i (with $1 \leq i, j \leq n$), $|\pi_{ij}|_A$ depends only on i and j.
3. The matrix A admits a *potential* if there exists a vector $v \in \mathcal{K}^n$ (called potential), such that for all $1 \leq i, j \leq n$ and for all paths π_{xy} from j to i, $|\pi_{ij}|_A = v_i v_j^{-1}$.
4. A FTEGM is *conservative* (resp. *admits the potential* v) if its multiplier matrix μ is conservative (resp. admits the potential v).
5. A FTEGM is *linearizable* if there exists a diagonal change of variables $Z_q(t) = v_q \times Z'_q(t)$, with $v_q \in \mathbb{R}^+_* \stackrel{\text{def}}{=} \mathbb{R}^+ \setminus \{0\}$, such that Z'_q satisfy min-plus linear recurrent equations; or equivalently and more formally, if three diagonal matrices $V_x = \mathrm{diag}(v_q, q \in \mathcal{X})$ and similarly V_u, V_y, are such that the entries of $C' = (V_y)^{-1} C V_x$, $A' = V_x^{-1} A V_x$, $B' = V_x^{-1} B V_u$, $D' = (V_y)^{-1} D V_u$ belong to $\mathbb{R}_{\min}[\delta]$.
6. A FTEGM with transfer $\mathcal{H}$ is *homogeneous* if there exist two vectors[6] $v_u \in (\mathbb{R}^+_*)^{\mathcal{U}}$, $v_y \in (\mathbb{R}^+_*)^{\mathcal{Y}}$ such that
$$\forall \lambda \in \mathbb{R}_{\min}, \quad \mathcal{H}(\lambda v_u + u) = \lambda v_y + \mathcal{H}(u) \ , \tag{4.1}$$
where $\lambda v + u$ denotes the vector signal $t \mapsto \lambda v + u(t)$.
7. A FTEGM is *trim* if each transition is structurally controllable and observable, i.e. if for each transition, there exists a path coming from at least one input, and there exists a path going to at least one output.

REMARK 4.2. 1. Clearly not all matrices admit a potential.
2. A FTEGM is linearizable if it reduces to an ordinary FTEG by a change of counting units.
3. The homogeneity property (4.1) extends the usual linearity relation (3.2b) which is the specialization of (4.1) to vectors v_u, v_y with all entries equal to 1.
4. The additivity axiom (3.2a) is automatically satisfied for FTEGM, since the equations (3.5) involve only the shifts and scaling operators (3.3), which are additive.

The main result of this paper is the following characterization, which is proved in Appendix 6.2 as a consequence of a more general lemma on potentials of matrices.

[5] We mostly use the semifield $(\mathbb{R}^+, \max, \times)$. However, it is instructive to note that the heart of the results (Lemma 6.1 below) holds in general semifields.

[6] Later on, we shall see that v_u and v_y are components of a potential v.

THEOREM 4.3. *Let $\mathcal{E}$ denote a FTEGM. The two following assertions are equivalent.*

1. *$\mathcal{E}$ is linearizable;*
2. *$\mathcal{E}$ has a potential.*

Moreover, the above assertions imply that

3. *$\mathcal{E}$ is homogeneous;*
4. *$\mathcal{E}$ is conservative.*

Conversely, if $\mathcal{E}$ is homogeneous, trim and explicit, then it is linearizable. If $\mathcal{E}$ is conservative and strongly connected, then it is linearizable.

From this theorem, the following result is clear.

COROLLARY 4.4. *A FTEGM reduces to a TEG by a change of counting units iff it has a potential.*

As it is well known [1, 5], autonomous TEGs reach a periodic regime after a finite time. This property being preserved by a change of counting units, we obtain as an immediate corollary of [1, Th. 3.28] and Theorem 4.3 a periodicity theorem for linearizable FTEGM.

COROLLARY 4.5 (CYCLICITY). *The counter functions of an* autonomous *linearizable FTEGM satisfy the following periodicity property:*

$$\exists T_0, c \geq 1, \quad \forall i \in \mathcal{Q} \cup \mathcal{P}, \exists \lambda_i, \quad \forall t \geq T_0, \quad Z_i(t+c) = \lambda_i c + Z_i(t) \ .$$

Moreover, for a strongly connected graph[7]*, the* periodic throughput λ_i *at node i is given by the expression*

$$v_i^{-1}\lambda_i = \min_{\mathcal{C}} \frac{\sum_{p\in\mathcal{C}} v_p^{-1} m_p}{\sum_{p\in\mathcal{C}} \tau_p} \ , \tag{4.2}$$

for any potential v that fits this linearizable FTEGM, where the minimum is taken over all the circuits $\mathcal{C}$ of the graph, and the sums are taken over all the places of the circuit $\mathcal{C}$. □

It is important to note that the terms at the right-hand side of (4.2) are *invariants* of the net. Equivalently, for all circuits $\mathcal{C}$, the vector indexed by $\mathcal{P}$, with entries v_p^{-1} for $p \in \mathcal{C}$, and $v_p = 0$ otherwise, is a P-semiflow [17].

PROPOSITION 4.6 (INVARIANTS). *Let v denote a potential and $\mathcal{C}$ a circuit of a FTEGM. For all markings m' reachable from the initial marking m, we have*

$$\sum_{p\in\mathcal{C}} v_p^{-1} m_p = \sum_{p\in\mathcal{C}} v_p^{-1} m'_p \ .$$

Proof. Let q denote a transition of $\mathcal{C}$. Let $q^- = q^{\text{in}} \cap \mathcal{C}$, $q^+ = q^{\text{out}} \cap \mathcal{C}$ denote respectively the places of the circuit upstream and downstream q. Ignoring the trivial case where $\mathcal{C}$ is a loop, we assume that $q^+ \neq q^-$. After firing transition q, we obtain the new marking $m'_{q^+} = m_{q^+} + \mu_{q^+q}$, $m'_{q^-} = m_{q^-} - \mu_{qq^-}^{-1}$, the markings of the other places of $\mathcal{C}$ being unaltered.

[7] The extension to the non strongly-connected case is straightforward.

Thus, the sum $\sum_{p\in\mathcal{C}} v_p^{-1} m_p$ increases by $v_{q^+}^{-1}\mu_{q^+q} - v_{q^-}^{-1}\mu_{qq^-}^{-1}$ which is zero precisely because v is a potential. □

The algorithm below allows us to compute a potential. It uses the *undirected* transition graph of A, $\mathcal{G}'(A) = (V_A, E_A)$, with set of vertices $V_A = \{1, \dots, n\}$, and set of edges[8] $E_A = \{\{i, j\} \mid A_{ij} \neq \varepsilon \text{ or } A_{ji} \neq \varepsilon\}$. The algorithm runs in time $O(n+m)$, where m is the number of arcs of $\mathcal{G}(A)$. It is based on the observation that the value of v at an arbitrary node j determines the value of v at all the successors and predecessors of j, using the relations $v_i = A_{ij}v_j$ and $v_i = A_{ji}^{-1}v_j$, when $A_{ij} \neq \varepsilon$ and $A_{ji} \neq \varepsilon$, respectively. Each iteration of the repeat loop explores a connected component of $\mathcal{G}'(A)$, and determines the value of v on it. The correctness of the algorithm follows readily from Lemma 6.1 and Remark 6.2 below. Practically, a classical convenient way to implement the graph exploration (while loop) is a depth first search (see [16, 2] and [12, Ch. 1]). It is easy to prove that the number of compatibility conditions to check is equal to the “cyclomatic number” of the undirected multigraph obtained from $\mathcal{G}(A)$ by disregarding the orientation of arcs.

Algorithm 1 Computing a Potential

INPUT: a $n \times n$-matrix A, with entries in a semifield $\mathcal{K}$.
OUTPUT: the potential of A in $\mathcal{K}^n$ if it exists, false otherwise.
All the nodes are initially non visited.
repeat
  mark an arbitrary non visited node r as visited, and assign an arbitrary value to v_r;
  while there is an edge $\{i, j\}$ of $\mathcal{G}'(A)$, with visited j, and unvisited i, **do**
    mark node i as visited,
    if $A_{ij} \neq \varepsilon$ **then**
      $v_i \leftarrow A_{ij}v_j$,
      mark arc (i, j);
    else
      $v_i \leftarrow A_{ji}^{-1}v_j$,
      mark arc (j, i);
    end if
  end while
  for all unmarked arcs (i, j), with visited ends **do**
    check the compatibility condition $v_i = A_{ij}v_j$,
    answer **false** if it is not satisfied,
    mark arc (i, j);
  end for
until all the nodes are visited.

[8]We denote edges of undirected graphs by unordered pairs $\{i, j\}$, as opposed to arcs of directed graphs denoted by ordered pairs (i, j).

EXAMPLE 4.7. We come back to the FTEGM depicted in Fig. 1. The multiplier matrix has the block partition

$$\mu = \begin{pmatrix} 0 & \mu_{\mathcal{QP}} \\ \mu_{\mathcal{PQ}} & 0 \end{pmatrix}, \quad \mu_{\mathcal{PQ}} \stackrel{\text{def}}{=} \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & u & y \end{array} \\ \begin{pmatrix} \cdot & \cdot & 3 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \end{array},$$

$$\mu_{\mathcal{QP}} \stackrel{\text{def}}{=} \begin{array}{c} \\ x_1 \\ x_2 \\ u \\ y \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array} \\ \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1/2 & 1 & \cdot & 1/2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \end{array}.$$

(the zero entries are represented by dots).

Setting $v_u = 1/3$, and visiting the nodes in the order $u, p_1, x_2, p_2, p_3, x_1, p_4, p_5, p_6, y$ Algorithm 1 gives

$$v^T = \begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & u & y & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array} \\ \begin{pmatrix} 1 & 1/2 & 1/3 & 1 & 1 & 1/2 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The three unmarked arcs (x_2, p_4), (x_2, p_2), (x_1, p_5) give the three following (satisfied) compatibility conditions: $v_{x_2} = (1/2)v_{p_4}$, $v_{x_2} = v_{p_2}$, $v_{x_1} = v_{p_5}$. Thus, v is a potential.

With the new variables $x'_1 = x_1,\ x'_2 = 2x_2,\ u' = 3u,\ y' = y$, the system admits the following representation

$$A' = \begin{pmatrix} \gamma^2\delta & \gamma\delta \\ \delta & \gamma^2\delta^3 \end{pmatrix}, \quad B' = \begin{pmatrix} \varepsilon \\ \gamma \end{pmatrix}, \quad C' = \begin{pmatrix} \gamma & \varepsilon \end{pmatrix}. \tag{4.3}$$

The corresponding linearized TEG is shown in Fig. 2. It is not difficult[9] to

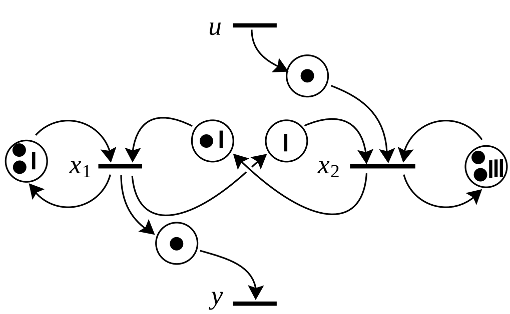

FIGURE 2. TEG obtained after the change of variables

compute the transfer series in the new system of coordinates:

$$H' \stackrel{\text{def}}{=} C'(A')^* B' = \gamma^3\delta(\gamma\delta^2)^* .$$

[9]See [9, 1] for more details on the rational identities used to obtain reduced expressions of the form (4.4).

In the original system of coordinates:

$$H = H'3 = \gamma^3\delta(\gamma\delta^2)^*3 = \bigoplus_{n\in\mathbb{N}} \gamma^{3+n}\delta^{1+2n}3 \ . \tag{4.4}$$

From (4.4), we get the explicit input-output relation

$$y(t) = [Hu](t) = \inf_{n\in\mathbb{N}} (3 + n + 3u(t-1-2n)) \ .$$

Finally, the application of the minimal mean-weight formula (4.2) gives the following expression of the periodic throughput, e.g. for transition x_2,

$$2\lambda_{x_2} = \min(2, 2/3, 1/2) = 1/2 \ , \tag{4.5}$$

which means that transition x_2 is asymptotically fired once every four time units in the autonomous regime.

We conclude by mentioning a case where there is no loss in considering the fluid approximation of a, originally discrete, TEGM.

PROPOSITION 4.8. *If the minimal integer valued*[10] *potential u of a TEGM, admitting a potential, and the normalized initial marking ν satisfy*

$$\forall q \in \mathcal{Q}, \forall p \in q^{\text{in}}, \quad \nu_p \in u_q\mathbb{N} \ , \tag{4.6}$$

then the earliest autonomous behavior of the TEGM coincides with that of its fluid version.

Proof. Performing the change of variables $Z_i' = u_i^{-1}Z_i$ in (2.2), we obtain

$$Z_q'(t) = u_q^{-1} \min_{p\in q^{\text{in}}} \left\lfloor u_q \left(u_q^{-1}\nu_p + Z'_{p^{\text{in}}}(t-\tau_p) \right) \right\rfloor \ . \tag{4.7}$$

Assuming by induction that $Z'_{p^{\text{in}}}(t-\tau_p)$ takes integer values, for all $p \in q^{\text{in}}$, the assumption that $u_q^{-1}\nu_p \in \mathbb{N}$ allows us to cancel the integer round-up at the right-hand side of (4.7), yielding

$$Z_q'(t) = \min_{p\in q^{\text{in}}} \left(u_q^{-1}\nu_p + Z'_{p^{\text{in}}}(t-\tau_p) \right) \ , \tag{4.8}$$

which shows that $Z_q'(t)$ is also integer. This shows by induction that Z' follows the fluid dynamics. □

Apart from these exceptional cases, it is not yet clear whether the more general expansion procedure developed by Munier admits a simple operatorial transcription, in the spirit of the $\mathcal{A}_{\min}[[\delta]]$ formalism presented here.

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[10] All the potentials of a connected FTEGM are proportional. Moreover, the ratios v_i/v_j are rational. Hence, the existence of a real valued potential guarantees the existence of a minimal integer valued potential.

5. CONCLUSION

To conclude, we would like to indicate the limitations of the approach presented here, and point out a few open questions. Theorem 4.3 provides a convenient way to reduce *fluid* TEGM admitting a potential to *fluid* TEG. One may therefore ask how coarse the fluid approximation can be. Since the dynamics of a (discrete) TEGM is obtained from its fluid approximation by taking down roundings, it is plain that the counter functions of the associated FTEGM dominate that of the original (discrete) TEGM. The equality case (Proposition 4.8) is exceptional. The discrete behavior may be arbitrarily far from the fluid one. Indeed a FTEGM with positive throughputs may have a deadlocked discrete version. For instance, the TEGM shown in Fig. 1 reaches a deadlock after transition x_1 is fired (since *two* tokens would be necessary in place p_4 to fire x_2) while the fluid version is live. Of course, the quality of the approximation increases when the values of the initial marking becomes large.

The results given here for event graphs have been (partly) extended to general Petri nets (see [8]). In this reference, one can see numerical experiments showing the quality of the fluid approximation.

Another open direction would be to treat general FTEGM, with no potentials. It is standard Bellman theory that general FTEGM recursions (see Eq. (3.1)) admit geometric growths or convergences. However, one cannot easily characterize the corresponding rational (transfer) series with the same degree of precision as transfer series of TEG are characterized [1].

Last, the presentation given here is not symmetric in counting and dating. A dual theory obviously exists, if one considers timing transformations of the form $Z'_q(t) = Z_q(v_q \times t)$, rather than counting transformations $Z'_q(t) = v_q Z_q(t)$. A more symmetric discussion, along the lines of [1, Chapter 5] in the TEG case, will be considered in a forthcoming study.

6. APPENDIX

In this appendix, we first give a lemma of general interest about matrices, and then we use it to prove Theorem 4.3.

6.1. POTENTIAL PROPERTIES OF MATRICES

We recall that the *classes* of a matrix A are, by definition, the strongly connected components of the transition graph of A, that a class is *initial* if there exists no other class upstream, and that it is *final* if there exists no other class downstream. A matrix with a single class is *irreducible*.

With a $n \times n$ matrix A with entries in a semifield $\mathcal{K}$, we associate the *symmetrized* matrix:

$$(A^{\mathrm{sym}})_{ij} \stackrel{\mathrm{def}}{=} \begin{cases} A_{ij} & \text{if } A_{ij} \neq \varepsilon, \\ A_{ji}^{-1} & \text{if } A_{ij} = \varepsilon \text{ and } A_{ji} \neq \varepsilon, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{6.1}$$

LEMMA 6.1. *Let $\mathcal{K}$ denote a semifield. Let $A \in \mathcal{K}^{n\times n}$. The following conditions are equivalent:*

1. *A^{sym} has a potential;*
2. *A has a potential;*
3. *there exists a vector v with non ε entries such that $A_{ij} \neq \varepsilon \Rightarrow v_i = A_{ij}v_j$;*
4. *there exists a vector v defined* only *on the initial and final classes such that for all paths $\pi_{\varphi\iota}$ from a vertex ι in an initial class to a vertex φ in a final class, $|\pi_{\varphi\iota}|_A = v_\varphi(v_\iota)^{-1}$;*
5. *there exist an (invertible) diagonal matrix $V \in \mathcal{K}^{n\times n}$ and a Boolean matrix B ($\in \{\varepsilon, e\}^{n\times n}$) such that $A = VBV^{-1}$.*

Moreover, if A is irreducible, the above conditions are equivalent to any of the following statements:

6. *for all circuits c, $|c|_A = e$;*
7. *A is conservative;*
8. *there exists a collection of paths $\overline{\pi}_{k1}$ (from 1 to $k = 2, \ldots , n$) such that*

$$\forall i, j, \quad A_{ij} \neq \varepsilon \Rightarrow A_{ij}|\overline{\pi}_{j1}|_A = |\overline{\pi}_{i1}|_A \quad ; \tag{6.2}$$

9. *property (6.2) holds independently of the choice of the paths $\overline{\pi}_{k1}$.*

In the irreducible case, these facts are essentially classical. See [11].

REMARK 6.2. The equivalence 1⇔2 allows us to consider A^{sym} rather than A. This is a useful trick, since we may always assume that A^{sym} is irreducible, and use the simpler characterizations of the second part of the lemma. Indeed, in general, A^{sym} is block-diagonal, with irreducible diagonal blocks. Thus, we have to find a potential for each irreducible block, separately. The more technical points 8 and 9, together with 1⇔2, justify Algorithm 1. Point 8 shows that it is enough to visit the graph in an arbitrary way starting from an arbitrary node (say 1): if the corresponding paths $\overline{\pi}_{k1}$ satisfy (6.2), we obtain a potential. Conversely, if this procedure fails for a special choice of paths, (9) implies that a potential does not exist.

REMARK 6.3. When the semifield is idempotent, the potential v is an eigenvector of the matrix A^{sym} associated with the eigenvalue e.

Proof of Lemma 6.1. The following implications are obvious

$$\begin{array}{ccccc} & 1 & \Rightarrow 7 \overset{\text{irred.}}{\Rightarrow} 9 \Rightarrow & 8 \\ & \Updownarrow & & \Downarrow \\ 4 \Leftarrow & 2 & \Longrightarrow & 3 \end{array}$$

the implication 7⇒9 holding only in the irreducible case.

3⇒5. We choose the diagonal matrix $V_{ii} = v_i$. Then $(V^{-1}AV)_{ij} = v_i^{-1}A_{ij}v_j = e$ or ε, hence, $B = V^{-1}AV$ is Boolean.

5⇒2. For all paths π_{ij}, $|\pi_{ij}|_{VBV^{-1}} = v_i v_j^{-1}$.

7⇒6. Let i be the initial point of c. Since both c and c^2 are paths from i to i, $|c|_A = |c|_A^2$, thus $|c|_A = e$.

6⇒7 (when A is irreducible). Let π, π' denote two paths from j to i. Since A is irreducible, there is a path π'' from i to j. Since $\pi\pi''$ and $\pi'\pi''$ are circuits, $|\pi\pi''|_A = |\pi|_A|\pi''|_A = |\pi'\pi''|_A = |\pi'|_A|\pi''|_A$, hence, $|\pi|_A = |\pi'|_A$.

4⇒2. For each vertex i, we choose an arbitrary vertex $\iota(i)$ in an initial class upstream i and we choose an arbitrary path $\pi'_{i\iota(i)}$. We set $u_i = |\pi'_{i\iota(i)}|_A v_{\iota(i)}$. Let $\pi''_{\varphi(i)i}$ denote a path from i to a vertex $\varphi(i)$ in a final class. The potential property of v restricted to the initial and final classes yields

$$\begin{aligned} v_{\varphi(i)} &= |\pi''_{\varphi(i)i}\pi_{ij}\pi'_{j\iota(j)}|_A v_{\iota(j)} = |\pi''_{\varphi(i)i}|_A|\pi_{ij}|_A u_j \ , \\ v_{\varphi(i)} &= |\pi''_{\varphi(i)i}\pi_{i\iota(i)}|_A v_{\iota(i)} = |\pi''_{\varphi(i)i}|_A u_i \ , \end{aligned}$$

hence, cancelling $|\pi''_{\varphi(i)i}|_A$, we get $|\pi_{ij}|_A u_i = u_j$. That is, u is a global potential. □

6.2. Proof of Theorem 4.3

i) *Potential equivalent to linearizable.* Let v denote a potential of the graph. Then, the fluid version of (2.1) becomes

$$Z_q(t) = \min_{p\in q^{\text{in}}} v_q v_p^{-1}\left(Z_p(t-\tau_p)+m_p\right) \tag{6.3}$$

$$Z_p(t) = v_p v_{p^{\text{in}}}^{-1} Z_{p^{\text{in}}}(t) \ , \tag{6.4}$$

hence the system becomes min-plus linear after the change of variables $Z'_i = v_i^{-1}Z_i$. The converse implication is obtained along the same lines.

ii) *Linearizable implies homogeneous.* Transforming the linearity property of TEGs (3.2b) by a diagonal change of variables, we obtain the homogeneity property (4.1).

iii) *Potential implies conservative.* This is the implication 2⇒7 of Lemma 6.1.

iv) *Homogeneous, trim and explicit implies the potential property.* We only consider the single input single output case (the general case being similar). Then, the transfer series defined by (3.6) is scalar, i.e. $H \in \mathcal{A}_{\min}[[\delta]]$. Consider the expansion

$$H = \bigoplus_{\tau\in\mathbb{N}} H_\tau\delta^\tau, \quad H_\tau \in \mathcal{A}_{\min} \ . \tag{6.5}$$

First, we note that if the system $u \mapsto y = Hu$ is (v_u, v_y)-homogeneous, then each map $H_\tau : \mathbb{R} \to \mathbb{R}$ is (v_u, v_y)-homogeneous. This can be seen by introducing the "Dirac function", $e(t) \stackrel{\text{def}}{=} e$ if $t = 0$, ε otherwise, and noting that for all $z \in \mathbb{R}$, $H(ze)(\tau) = H_\tau(z)$.

Next, we note that the (v_u, v_y)-homogeneity of the map $H_\tau : \mathbb{R} \to \mathbb{R}$ readily implies that

$$\forall z \in \mathbb{R}, \quad H_\tau(z) = \nu_\tau + (v_y)(v_u)^{-1} z \ , \tag{6.6}$$

with $\nu_\tau = H_\tau(0)$.

Last, the path interpretation of the transfer (3.8) gives $H_\tau = \bigoplus_\pi \gamma^{\nu_\pi} |\pi|_\mu$, where the sum is taken over all the paths π from the input to the output, with sum of the holding times τ. Note that the sum is indeed finite, since we assume that there are no circuits with zero holding times. More explicitly,

$$\forall z \in \mathbb{R}, \quad H_\tau(z) = \min_\pi \left(\nu_\pi + |\pi|_\mu z\right) . \tag{6.7}$$

A necessary condition for (6.6) and (6.7) to coincide is obviously that, $\forall \pi,\ |\pi|_\mu = v_y(v_u)^{-1}$. Since the graph is trim, the input transition is the only initial class and the output transition is the only final class. The conclusion follows from Lemma 6.1, part 4⇒2.

v) *Conservative and strongly connected imply linearizable.* This is the implication 7⇒2 of Lemma 6.1. This concludes the proof of Theorem 4.3. □

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