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Minplus Homogeneous Dynamical Systems

Growth rate, Eigenvalues and Traffic Applications

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Good Retirement Geert Jan !!

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Definitions

Minplus homogeneous dynamical systems:

$$x^{k+1} = f(x^k), \text{ with } f : \mathbb{R}^n_{\min} \mapsto \mathbb{R}^n_{\min} : \; f(\lambda \otimes x) = \lambda \otimes f(x) \, .$$

Growth rate $\chi \in \mathbb{R}_{\min}$:

$$\chi = \lim_k x_i^k / k, \quad \forall i = 1, \cdots , n \, .$$

Eigenvalues $\lambda \in \mathbb{R}_{\min}$:

$$\exists x \neq \varepsilon : f(x) = \lambda \otimes x.$$

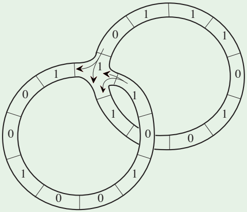
Problems and Applications

Questions:

$$?\exists\chi, \quad ,?\exists\lambda, \quad ?\chi=\lambda\ .$$

TRUE when f is monotone and $\mathcal{G}(f)$ strongly connected.

Traffic Applications (f homogeneous not monotone):



Canonical form of Homogeneous Systems

The dynamics $x^{k+1} = f(x^k)$ is equivalent to

$$\begin{cases} x_1^{k+1}/x_1^k = f_1(x^k)/x_1^k, \\ x_i^{k+1}/x_1^{k+1} = f_i(x^k)/f_1(x^k), \quad i = 2, \cdots, n, \end{cases}$$

using the homogeneity it can be written :

Dynamics Canonical Form

$$\begin{cases} \Delta^k = h(y^k), \\ y^{k+1} = g(y^k), \end{cases}$$

with $\Delta^k \triangleq x_1^{k+1}/x_1^k$, $y_{i-1}^k = x_i^k/x_1^k$ and $g_{i-1} = f_i/f_1$ for $i = 2, \cdots, n$.

Growth rate

As soon as the y^k belong to a bounded closed (compact) set for all k, the set of measures:

$$\left\{P_{y^0}^N = \frac{1}{N}\left(\delta_{y^0} + \delta_{g(y^0)} + \cdots + \delta_{g^{N-1}(y^0)}\right),\ N \in \mathbb{N}\right\},$$

is tight. Therefore we can extract convergent subsequences which converge towards invariant measures Q_{y^0}.

Applying the ergodic theorem to the sequence $(y^k)_{k\in\mathbb{N}}$:

Growth Rate Existence

$$\chi = \frac{1}{N}(x_1^N - x_1^0) = \lim_N \frac{1}{N}\left(\sum_{k=0}^{N-1} h(y^k)\right) = \int h(y) dQ_{y^0}(y),\ Q_{y^0} \text{ a.e.}$$

Remarks on Growth rate Existence

It would be very useful to prove that the limit exists for sequence starting from y^0.

1. A priori homogeneous systems have not the uniform continuity property necessary to prove the convergence of the Cesaro means for y^0.
2. In the case where the compact set is finite, we can apply the ergodicity results on Markov chains with a finite state number to show the convergence of $P^N_{y^0}$ towards Q_{y^0} which proves the convergence of the Birkhoff average for the sequence starting from y^0.

Non Everywhere Convergence of Birkhoff Averages

$$f : x \in \mathbb{T}^1 \rightarrow 2x \in \mathbb{T}^1 \text{ with: } x^0 = 0.10\overset{2}{0}111\overset{4}{1}0000000\overset{8}{0}\cdots$$

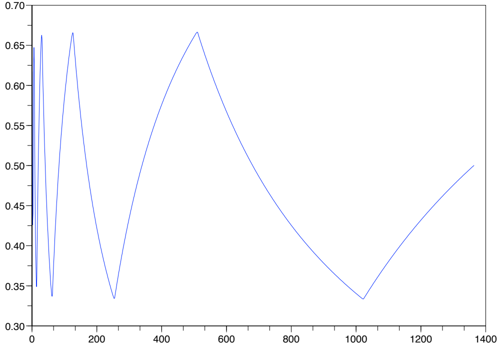


Figure: Plot of $S(n)$ with: $S(n) \triangleq \frac{1}{n}\sum_{k=0}^{n-1} x^k$.

Eigenvalue of Homogeneous Systems

The eigenvalue problem a function $f : \mathbb{R}^n_{\min} \mapsto \mathbb{R}^n_{\min}$ can be formulated as finding $x \in \mathbb{R}^n_{\min}$ non zero, and $\lambda \in \mathbb{R}_{\min}$ such that:

$$\lambda \otimes x = f(x) .$$

Since f is homogeneous, we can suppose without loss of generality that if x exists then $x_1 \neq \varepsilon$ and we have the:

Eigenvalue Canonical Form:

$$\begin{cases} \lambda &= h(y) , \\ y &= g(y) , \end{cases}$$

with $y_{i-1} = x_i / x_1$, $h(y) = f_1(x)/x_1$ and $g_{i-1} = f_i / f_1$ for $i = 2, \cdots, n$.

Eigenvalue Existence

Eigenvector Existence

The existence of eigenvalue is reduced to the existence of the fixed point of g which gives an eigenvector.

Standard Examples

1. f is a finite Markov chain transition operator.
2. f is affine in standard algebra with $dim(ker(f' - I_d)) = 1$.
3. f is minplus linear.
4. f is a dynamic programming function associated to a stochastic control problem.
5. f is a dynamic programming function associated to a stochastic game problem.

Affine Example with $dim(ker(f' - I_d)) = 1$.

With standard notations we have to solve:

$$\lambda + x = Mx + b,\ M\mathbf{1} = \mathbf{1},\ \text{ Eigenvalue 1 simple .}$$

Using the variable change $z = Px$ with:

$$z = \begin{bmatrix} x_1 \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ -1 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & \cdot & \cdot & 1 \end{bmatrix} x \ .$$

The system $\lambda P\mathbf{1} + z = PMP^{-1}z + Pb$ has a block triangular form $PMP^{-1} = \begin{bmatrix} 1 & c \\ 0 & N \end{bmatrix}$ (thanks to the homogeneity $M\mathbf{1} = \mathbf{1}$), N has not the eigenvalue 1 (since 1 is a simple eigenvalue of PMP^{-1}) and therefore g has a unique fixed point.

Tent Example

Let us consider the homogeneous system:

$$\begin{cases} x_1^{k+1} = x_2^k \ , \\ x_2^{k+1} = (x_2^k)^3/(x_1^k)^2 \oplus 2(x_1^k)^2/x_2^k \ . \end{cases}$$

We have $h(y) = y$ and $g(y) = y^2 \oplus 2/y^2$ (g is the tent transformation which is chaotic).

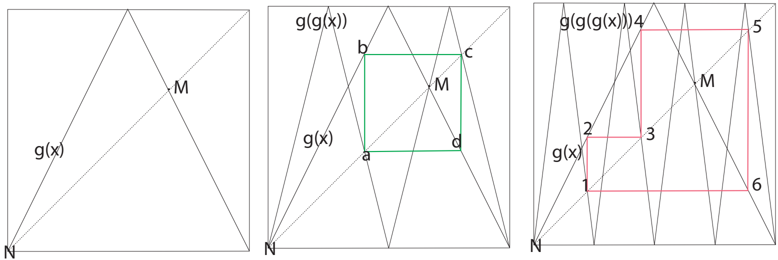


Figure: Tent transformation and its iterates.

$\chi \neq \lambda$

The eigenvalues are $\lambda = y$ solution of $y = y^2 \oplus 2/y^2$ that is:

$$\lambda \in \left\{0, \frac{2}{3}\right\}.$$

1. Starting from $y0 = \frac{2}{5}$, the trajectory is periodic of period 2. The invariant measure is $Q_{y^0} = \frac{1}{2}(\delta_{\frac{2}{5}} + \delta_{\frac{6}{5}})$, therefore:

$$\chi = \frac{4}{5}, \quad Q_{y^0} \text{ a.e.}$$

2. The tent transformation admits the uniform law as invariant measure, therefore:

$$\chi = \int_0^1 y dy = \frac{1}{2}, \quad \text{a.e. for the Lebesgue measure.}$$

2 Circular Roads with 1 junction

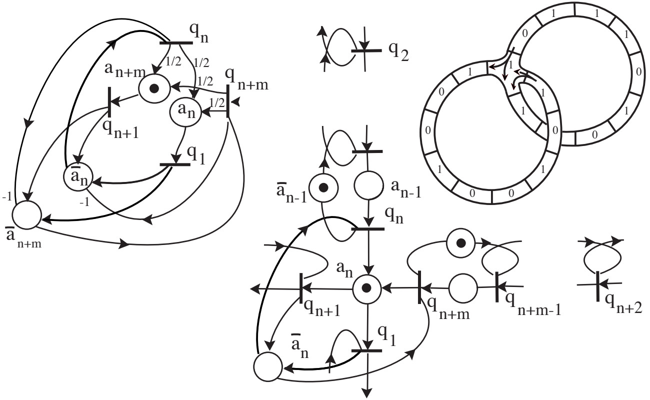


Figure: A junction with two circular roads cut in sections (top-right), its Petri net simplified modeling (middle) and the precise modeling of the junction (top left).

Dynamics

The general Petri net equation:

$$\min_{p \in q^{in}} \left\{ a_p + \sum_{q \in p^{in}} m_{pq} q^{k-1} - \sum_{q \in p^{out}} q^k \right\} = 0, \ \forall q \in \mathcal{Q}, \ \forall k,$$

does not define completely the dynamics.
We precise the dynamics by giving the turning probability (1/2) and the right priority to enter in the junction.

$$\begin{cases} q_i^{k+1} = a_{i-1} q_{i-1}^k \oplus \bar{a}_i q_{i+1}^k, \ i \neq 1, n, n+1, n+m, \\ q_n^{k+1} = \bar{a}_n \frac{q_1^k q_{n+1}^k}{q_{n+m}^k} \oplus a_{n-1} q_{n-1}^k \ , \\ q_{n+m}^{k+1} = \bar{a}_{n+m} \frac{q_1^k q_{n+1}^k}{q_n^{k+1}} \oplus a_{n+m-1} q_{n+m-1}^k \ , \\ q_1^{k+1} = a_n \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_1 q_2^k \ , \\ q_{n+1}^{k+1} = a_{n+m} \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_{n+1} q_{n+2}^k \ . \end{cases}$$

Increasing trajectory property

Theorem

The trajectories of the states $(q_i^k)_{k\in\mathbb{N}}$, starting from 0, are nondecreasing for all i.

Proof by induction. For q_n:

If $q_n^{k+1} = a_{n-1} q_{n-1}^k$

$$\Rightarrow q_n^{k+1} \geq a_{n-1} q_{n-1}^{k-1} \geq f_n(q^{k-1}) = q_n^k \ .$$

If $q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$,

$$\Rightarrow q_n^{k+1} \geq \bar{a}_n q_n^k q_1^k q_{n+1}^k / \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1}$$

since $q_{n+m}^k \leq \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1} / q_n^k$

$$\Rightarrow q_n^{k+1} \geq q_n^k$$

Distances between states stay bounded

Theorem

The distances between any pair of states stay bounded:

$$\exists c_1 : \sup_k |q_i^k - q_j^k| \leq c_1, \forall i, j.$$

Moreover:

$$\forall T, \exists c_2 : \sup_k |q_i^{k+T} - q_i^k| \leq c_2 T, \forall i.$$

Existence of the Growth Rate

Theorem

There exists an initial distribution on $(q_j^0/q_1^0)_{j=2,n+m}$, the Kryloff Bogoljuboff invariant measure, such that the average flow

$$\chi = \lim_k q_i^k/k, \ \ \forall i \ ,$$

exists almost everywhere.

Eigenvalue Formula

The eigenvalue problem can be solved explicitly.

Theorem

The nonnegative eigenvalues λ *are solutions of:*

$$\top\left\{-\lambda, \perp\left\{(1-\rho)\,d-\lambda, \frac{1}{4}-\lambda, r-(1-\rho)\,d-(2r-1+2\rho)\,\lambda\right\}\right\}=0$$

with $N = n + m$, $\rho = 1/N$, $r = m/N$, d *the car density.*

$N >> 1,\ r > 1/2$

$$\lambda \simeq \max\left\{0,\ \min\left\{d, \frac{1}{4}, \frac{r-d}{2r-1}\right\}\right\}.$$

Difference Between Eigenvalue and Growth Rate

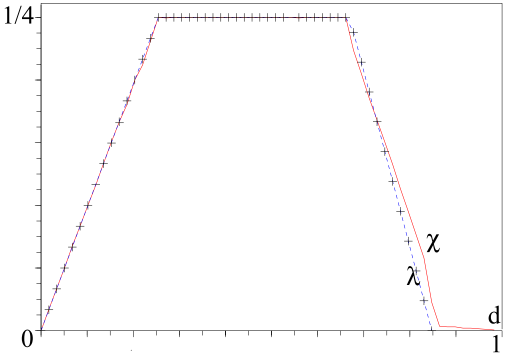
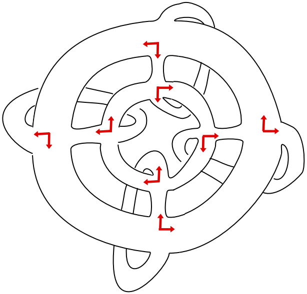


Figure: The traffic fundamental diagram $\chi(d)$ when $r = 5/6$ (continuous line) obtained by simulation and its comparison with the eigenvalue $\lambda(d)$.

Phases

1. *Free moving:* When the density is small, $0 \leq d \leq \alpha$ with $\alpha = \frac{1}{4(1-\rho)}$, after a finite time, all the cars move freely.
2. *Saturation:* When $\alpha \leq d \leq \beta$ with $\beta = \frac{1}{2}\frac{r+1/2-\rho}{1-\rho}$ the junction is used at its maximal capacity without being bothered by downstream cars.
3. *Recession:* When $\beta < d < \gamma$ with $\gamma = \frac{r}{1-\rho}$ the crossing is fully occupied but cars sometimes cannot leave it because the roads where they want to go are crowded. When $\gamma < \beta$, on the interval $[\gamma, \beta]$ three eigenvalues exist. In this case the system is in fact blocked.
4. *Blocking:* When $\gamma \leq d \leq 1$, the road without priority is full of cars, no car can leave it and one car wants to enter.

Extension to Regular Towns



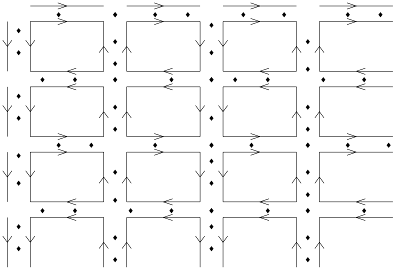


Figure: Roads on a torus of 4×2 streets with its authorized turn at junctions (left) and the asymptotic car repartition in the streets on a torus of 4×4 streets obtained by simulation.

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