Fundamental Traffic Diagrams of Elementary Road Networks.

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***Abstract*— The fundamental traffic diagram gives the relation between the flow and the density of vehicles on a road. This diagram has been obtained for one circular road in the deterministic and stochastic cases. In this paper we revisit the simplest example of two roads and one crossing given in [2] and we study the interaction between crossings on a system of four roads with two crossings. After showing numerically the existence of an average flow, we compute the fundamental diagram of the system and we study its various phases. We compare the fundamental diagrams obtained with different crossing policies (right priority, open-loop and feedback traffic light). We show that the "right priority" policy blocks the system at a specific vehicle density. This jam is explained by the appearance of a road circuit full of vehicles. Finally we show the important influence of the crossing design on the fundamental diagram.**

I. Introduction

The fundamental traffic diagram giving the relation between the flow and the density of vehicles on a road has been obtained theoretically and by simulation for one road in [7], [12], [5], [15], [3], [8], [6]. This diagram has also been obtained, numerically in [2], for a system of two roads with a crossing (in the deterministic case with a given turning policy). In this reference, the different phases of the system are analyzed. This kind of systems (in the stochastic case without turning possibility) have been also studied in [9], [10], [11]. Often, in statistical physics, cellular automaton models are used to derive this macroscopic law from microscopic models [6], [13],[14].

In this paper, we study the fundamental traffic diagram for a system of four roads and two crossings in order to show the effect of the interaction between the two crossings. We adopt the same approach as the one used in [3] and [2]. From a Petri net model, we derive a dynamics using linear operators in standard and minplus algebras. By simulations we show the existence of an average flow which is independent of the initial position of the vehicles. In the fundamental diagram, clearly, appear three phases depending on the vehicle density. The high density phase is obtained as soon as there is enough vehicles to form a road circuit full of vehicles. Then we study the influence on the fundamental diagram of three crossing control policies : right-priority, openloop traffic light control, and feedback traffic light. Finally we study the influence of crossing design. On the simplest system it appears a large improvement of the average flow at the medium density by the addition of one vehicle place in the junction.

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II. Minplus Algebra

In this section we recall the minplus algebra results that we will use in the traffic modeling sections. The reader can consult [1] for more details. The structure $\mathbb{R}_{\min} = \{\{\mathbb{R} \cup \{+\infty\}, \oplus, \otimes\}$ is defined by the set $\mathbb{R} \cup \{+\infty\}$ endowed with two operations: $\min$ (denoted by $\oplus$, called minplus *sum*) and $+$ (denoted by $\otimes$, called minplus *product*). The element $\varepsilon = +\infty$ is the *zero* element: $\varepsilon \oplus x = x$, it is absorbent: $\varepsilon \otimes x = \varepsilon$. The element $e = 0$ is the *unit* element: $e \otimes a = a$.

The principal differences with the standard algebra are the *idempotency* and the nonsimplification of addition: $a \oplus a = a$, and $a \oplus b = c \oplus b \nRightarrow a = c$. This algebra is called *minplus* algebra. From the minplus structure on scalars, we induce an idempotent semiring structure on the set of square matrices with the element-wise minimum for addition, and product operation defined by

$$(A \otimes B)_{ij} = \min_k (A_{ik} + B_{kj}).$$

The zero element and the unit element are still denoted by ε and e. We associate to a square matrix A a precedence graph $\mathcal{G}(A)$ whose nodes correspond to the columns of the matrix A and whose arcs correspond to nonnull entries (the weight of an arc (i, j) being the nonnull entry A_{ji}). We define the weight of a path p, denoted by $|p|_w$, the minplus product of weights of the arcs composing the path (i.e. the standard sum of its weights). The number of arcs of a path p is denoted by $|p|_l$. We recall the following principal results discusssed in details in [1].

Theorem 1: Given A a $m \times m$ minplus matrix, if the weights of all the circuits of $\mathcal{G}(A)$ are positive, then the equation $X = A \otimes X \oplus B$ admits a single solution: $X = A^* \otimes B$ where:

$$A^* = \bigoplus_{n=0}^{\infty} A^n = \bigoplus_{n=0}^{m-1} A^n$$

Theorem 2: If the associated graph $\mathcal{G}(A)$ with the minplus matrix A is strongly connected, then the matrix A has a unique eigenvalue $\lambda \in \mathbb{R}_{\min}$:

$$\exists X \in \mathbb{R}^n_{\min} : A \otimes X = \lambda \otimes X \text{ with } \lambda = \min_{c \in \mathcal{C}} \frac{|c|_w}{|c|_l}.$$

where $\mathcal{C}$ is the set of the circuits of $\mathcal{G}(A)$.

Theorem 3: If the graph $\mathcal{G}(A)$ associated with the minplus matrix A is strongly connected, then the linear miplus dynamics defined by: $X_{n+1} = A \otimes X_n$ is asymptotically periodic, i.e.

$$\exists T, K, \lambda : \forall k \geq K : A^{k+T} = \lambda^T \otimes A^k.$$

III. Continuous Timed Petri nets

A continuous Petri net is similar to a conventional Petri net but fluids composed of molecules circulate instead of tokens. Mathematically this means that the numbers of tokens which are integers, in a conventional Petri net, becomes fluid volumes and therefore are real numbers. Moreover the transition firings can be seen now as chemical reactions consuming available necessary molecules and producing other molecules.

A continuous timed Petri net is a graph with two kinds of nodes (*transitions* and *places*), and two kinds of arcs (*production arcs* and *synchronization arcs*). We denote by $\mathcal{Q}$ (with Q elements) the set of transitions, and by $\mathcal{P}$ (with P elements) the set of places. Production arcs go from transitions to places, and synchronization arcs go from places to transitions.

With a Petri net, we associate a minplus matrix $P \times Q$ denoted by D, called *synchronization matrix*, and two standard matrices the $Q \times P$ denoted $H(\delta)$ called *production matrix* and the $P \times Q$ matrix M called *consumption matrix*[1].

The synchronization matrix is defined by $D_{pq} = a_p$ if there exists an arc from $p \in \mathcal{P}$ to $q \in \mathcal{Q}$ and $D_{pq} = \varepsilon$ if not, where a_p is the initial amount of fluid in place p still shown graphically by the tokens in the places but which is now a real number. The production matrix is defined by a delay operator $H_{qp}(\delta) = h_{qp}\delta^{\tau_p}$ if there exists an arc from $q \in \mathcal{Q}$ to $p \in \mathcal{P}$ and $H_{qp}(\delta) = 0$ otherwise, where h_{qp} is the multiplicity of the arc (q, p), it determines the number of molecules produced in p by one firing of q (explained later). The number τ_p is the minimum sojourn time of molecules in the place p, called *temporization*, represented graphically by sticks in the places. A delay operator acts on series, it is defined by $h\delta^{\tau} : (X_n)_{n\in\mathbb{Z}} \mapsto h(X_{n-\tau})_{n\in\mathbb{Z}}$. The consumption matrix M is defined by $M_{pq} > 0$ if there exists an arc from p to q, and $M_{pq} = 0$ if not. The number $M_{pq} > 0$ is the multiplicity of the arc (p, q), it determines how many molecules are consumed in the place p by one firing of the transition q. Thus, a Petri net is characterized by the 5-uple : $\{\mathcal{P}, \mathcal{Q}, H(\delta), M, D\}$.

We say that a Petri net is *deterministic* if from each place leaves exactly one downstream arc (one nonnull entry in each line of the consumption matrix). If there is a place from which leave more than one arc it is called *nondeterministic*. Dynamics of Petri nets is constrained by what is called the *transition firings*. At a given time, a transition q can fire if in each upstream place, there is some molecules of fluid whose sojourn time exceeds the minimum sojourn time in the place. We denote by q^{in} (resp. q^{out}) the set of places upstream (resp. downstream) of the transition q. One firing consumes (from each place $p \in q^{in}$) M_{pq} molecules and produces (to each place $p \in q^{out}$) h_{qp} molecules.

In the case of nondeterministic Petri nets, transition firing does not completely define the dynamics of the Petri net, because transitions downstream of some places would be in conflict for molecules consumption. In this case, we must give consumption rules, for example by defining an order of priorities for transitions in conflict, or by imposing proportions of consumption. As soon as this kind of conflict is resolved, the Petri net becomes deterministic. Continuous Petri nets have dynamics given by the following theorem which defines uniquely or gives constraints to the firing trajectories X.

Theorem 4: Denoting by $X = (X^q)_{q\in\mathcal{Q}}$ the vector of the sequences $X^q = (X^q_n)_{n\in\mathbb{Z}}$ where the entry X^q_n denotes the cumulated firing number of the transition q up to time n, these firing trajectories satisfy [2] :

$$0 = \{X(H(\delta) - M^t)\} \otimes D \ .$$

Moreover the firing trajectories of deterministic Petri nets are defined uniquely by :

$$X = \{XH'(\delta)\} \otimes D' \ ,$$

with $H'(\delta)_{qp} = H(\delta)_{qp}/M_{pq'(p)}$ and $D'_{pq} = a'_p \triangleq a_p/M_{pq'(p)}$ (on the non null arcs) where $q'(p)$ denotes the index of the nonnull entry of the line p of M.
These equations tell that, at each time n, upstream each transition there is at least one place which has no available token (be in the place enough time to be used).

A. Event graphs

Deterministic Petri nets for which each place has only one upstream production arc of multiplicitiy 1 are called *event graphs*. The dynamics of an event graph is linear in minplus algebra : $X = X \otimes A(\delta)$ where $A(\delta)_{qq'} = a_p \otimes \delta^p$ and where p is the single place connecting q to q'. The following theorem is a corollary of Theorem 2 (see [1]).

Theorem 5: For a strongly connected event graph, the throughput defined by $\lambda = \lim_n X^q_n/n$ is independent of q, and it is equal to :

$$\lambda = \min_{c\in\mathcal{C}} \frac{|c|_a}{|c|_t},$$

where $\mathcal{C}$ is the set of circuits of the event graph, $|c|_a$ is the total initial amount of fluid in the circuit c, and $|c|_t$ is the total "minimal sojourn time" of the places in the circuit c.

B. Undiscounted Petri nets

If we impose consumption proportions by downstream transition of places, the dynamic of the corresponding nondeterministic continuous Petri net can be interpreted in term of optimal stochastic control [4].

We call *routing policy* ρ a constraint on the nondeterministic continuous Petri net imposing the proportion $\rho_{pq'}$ (with $\sum_{q'} \rho_{pq'} = 1$) of molecules leaving a place p to go to transition q'. The nondeterministic continuous Petri net for which the routing policy ρ_{pq} satisfies

$$\sum_q N^p_{qq'} = 1, \ \ \forall p, \ \ \text{with } N^p_{qq'} \triangleq h_{qp}\rho_{pq'}/M_{pq'},$$

[1] Sometimes we call consumption arcs the arcs corresponding to a negative production which are arcs from transitions to places. They must no be confused with the arcs of the matrix M which goes from places to transitions.

[2] if f is a nonlinear function, then $X = (X\delta)f$ means $(X_{n+1} = f(X_n))_{n\in\mathbb{N}}$

are called *undiscounted Petri nets*. The set $\{N^p, p \in \mathcal{P}\}$ can be seen as a set of stochastic transition matrices controlled by p.

Denoting by $\odot$ the contraction operation defined by :

$$(A \odot B)_{q'} = \bigoplus_p \{A_{pq'} \otimes B_{pq'}\}$$

for any couple of $Q \times P$ matrices A and B, the dynamic is given by the following theorem (see [4]).

Theorem 6: The undiscounted Petri nets have firing sequences satisfying

$$X = \{XH''(\delta)\} \odot D'' \ ,$$

with $H''(\delta)_{qpq'} = N^p_{qq'}\delta^{\tau_p}$ and $D''_{pq'} = a_p \rho_{pq'}/M_{pq'}$ the *initial weighted fluid* amount. The throughput λ is then the optimal average cost by unit of time of an undiscounted stochastic control problem :

$$\lambda = \min_{f \in \mathcal{F}} \frac{|f|_a}{|f|_t} \ ,$$

where : – $\mathcal{F}$ denotes the possible final classes of the Markov chains controlled by the set of feedback $s : \mathcal{Q} \mapsto \mathcal{P}$ (consisting to choose one place upstream each transition), – $|f|_a$ denotes the average initial weighted fluid amount in the final class f, – $|f|_t$ the average "minimum sojourn time" in the final class f (where the average is in the sense of the invariant probability of the final class f).

IV. Fundamental Diagrams for a System of Two Circular Roads and One Crossing

The fundamental diagram for one circular road with a speed reducer has been solved analytically in [2]. We recall here the fundamental diagrams obtained experimentally in [2] for a system of two circular roads and a crossing.

A. The Model

We consider a system of two one-way circular roads and a crossing, as one can see it on the left of Figure 1. Vehicles move without overtaking. The crossing circulation rule is "priority to the right". Same proportions of the cars leaving the junction go to south and to west. The Petri net model is given on Figure 2.

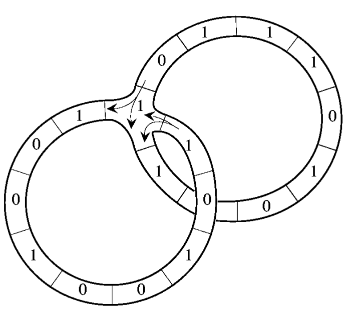

Fig. 1. Two circular roads and one crossing

Each road is cut in sections, the crossing is considered as a specific section. Each section can contain at most one car. One vehicle occupies one section. If the occupied section is not a crossing input, and if the section ahead is empty, then, the vehicle moves to the section ahead. A vehicle in a section connected to a crossing enters in the crossing if the crossing if free and if it is on the priority road. If the vehicle is on the non priority road it enters only if the crossing is free and if there is no vehicle on the priority road that wants enter. In the other cases the vehicle doesn't move.

We call density d the ratio p/m of the number of vehicles p by the number of sections m. We denote by X^q_n the total number of transition firings up to time n at transition q and we call flow the quantity $f = \frac{1}{m} \lim_{n\to+\infty} \frac{X^q_n}{n}$. This number is independent of q for connected graphs. The fundamental diagram gives the relation between f and d. On the Petri net

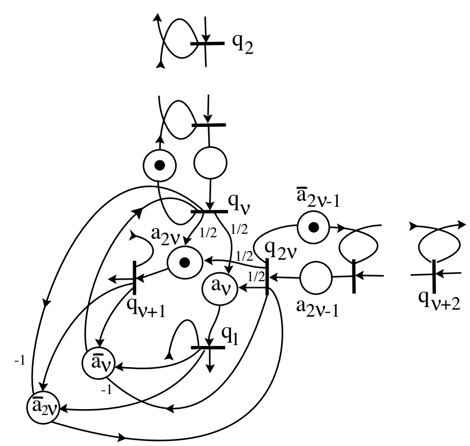

Fig. 2. Two circular roads and one crossing

of Figure 2, each road is made of ν sections and each section q is represented by two places a_q and $\bar{a}_q$. For road sections q, if $a_q = 0$ then $\bar{a}_q = 1$ and the section is free. If $a_q = 1$ then $\bar{a}_q = 0$ and there is a car in the section. For the crossing section, if $a_\nu = 0$ and $a_{2\nu} = 0$ then $\bar{a}_\nu = 1$ and $\bar{a}_\nu = 1$ and the crossing is free. Transition q corresponds to the access permission in section q. The crossing is a particular section represented by four places ($a_\nu, a_{2\nu}, \bar{a}_\nu, \bar{a}_{2\nu}$), with two entries ($q_\nu$, $q_{2\nu}$) and two exits (q_1, $q_{\nu+1}$).

The multiplicities $1/2$ express the fact that half of the vehicles entering to the crossing goes straight, while the other half turns.

Thanks to the multiplicities -1 we can count the authorization to enter in the crossing. For example a token in the place $\bar{a}_\nu$ gives an authorization to a vehicle from the north to enter in the crossing. The total number of north authorizations to enter is equal to the total number of vehicles having left the crossing minus the total of east authorizations given.

The vehicles are discrete objects (a half of a car has no meaning). But nevertheless we will approximate the traffic of discrete object by a fluid[3].

If we denote by X_n^q the firing number of the transition q up to time n, then the continuous dynamics written in minplus algebra, is (see [2]) :

- On the roads, the total number entered in the section q is equal to the minimum between : – the number (X_{q-1}) entered in section $q-1$ plus the number (a_{q-1}) of vehicle at initial time in this section, – the number (X_{q+1}) having left section q because they have entered in section $q+1$ plus the number ($\bar{a}_q$) of free places in section q at initial time :

$$X^q/\delta = a_{q-1}X^{q-1} \oplus \bar{a}_q X^{q+1}, \quad (1)$$
$$q \in \{2, \ldots, \nu-1, \nu+2, \ldots, 2\nu-1\} . \quad (2)$$

- On the crossing entries, the total number of entered vehicles is the minimum between the number of vehicles that want enter and the authorization of entering as discussed previously :

$$X^\nu/\delta = \bar{a}_\nu X^1 X^{\nu+1}/X^{2\nu} \oplus a_{\nu-1}X^{\nu-1} , \quad (3)$$
$$X^{2\nu}/\delta = \bar{a}_{2\nu} X^1 X^{\nu+1}/(X^\nu/\delta) \oplus a_{2\nu-1}X^{2\nu-1} . \quad (4)$$

- On the crossing exits the total number having left the crossing by the South [resp. West] is the minimum between half of the one entered in the crossing and the number having left the section after the crossing in the south [resp. West] direction :

$$X^1/\delta = a_\nu \sqrt{X^\nu X^{2\nu}} \oplus \bar{a}_1 X^2 , \quad (5)$$
$$X^{\nu+1}/\delta = a_{2\nu} \sqrt{X^\nu X^{2\nu}} \oplus \bar{a}^{\nu+1} X^{\nu+2} . \quad (6)$$

In these equations, the minplus division "/" is the standard subtraction, the minplus square is the division by 2 in standard algebra and δ is the delay operator acting on series. This system of equations is implicit but triangular, thus the trajectory is well defined, and the system can be written: $X = (XH(\delta)) \otimes D$. See the small example written in standard algebra shown in the appendix B, or [2] for a better understanding of the syntax of these equations.

B. Numerical Results

In Figure 3, we show the fundamental diagrams for different road section number ratios (from 1 to 10). See more comments and details in [2]. We can justify the existence of the average flow theoretically, only in the simplest case given in the appendix A. In this case, after some algebraic manipulation we can reduce the computation of the average flow to the throughput of undiscounted Petri nets (Theorem 6).

The general case can be written $X = f(X, \delta)$ with $f(X, \delta) = (XH(\delta)) \otimes D$ homogeneous of degree 1 in X (that is $f(\lambda \otimes X, \delta) = \lambda \otimes f(X, \delta)$) and nonpositive (there exists $X \geq 0$ such that $f(X, \delta) < 0$) is not reducible easily to dynamic programming problem. And the corresponding generalized eigenvalue problem : find λ and X nontrivial such that $X = f(X, \lambda)$ is still an open question on which we are working.

[3]The fluid approximation represents well the asymptotic discrete systems when we increase a scale factor (total number of sections with preservation of proportion of section numbers in the different roads). The fluid approximation dynamics is simpler : no rounding operator appears in the equations. See [2] for the dynamics of the discrete systems.

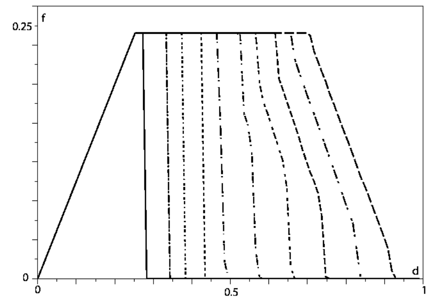

Fig. 3. Fundamental Diagrams for two roads and a crossing with different ratios of the section numbers in the two roads.

V. Four roads with two crossings

To have a better understanding of the inter blocking of the crossings, we consider in this section the case of four roads and two crossings as shown on Figure 4. It is supposed that vehicles move in the same way as in the preceding section. The "priority to the right" in this case implies that there is two priority roads. We show the existence of the fundamental diagram for this system and discuss its various phases.

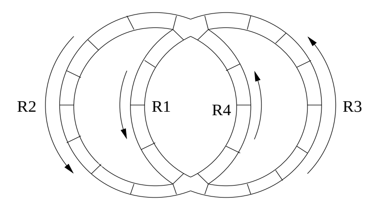

Fig. 4. Four roads with two crossings.

A. Minplus and Petri net modeling

The system is modelled by the Petri net shown on Figure 5. The roads are numbered as indicated on this figure. As in the preceding section, the model can be written by combining operators of standard algebra with those of minplus algebra. We denote by q_{ij} the jth transition ($j \in \{1, \cdots \nu\}$) of the road i, and by X_n^{ij} the number of firings of the transition $q_{(i,j)}$ up to time n. Using the minplus operators, the dynamics of this system is well defined.

In this case the system can be written also $X = (XH(\delta)) \otimes D$, but we will not explicit the operators H and D.

B. Global Fundamental Diagram

Simulations are done using the MaxPlus toolbox of Scilab [16]. As in the case of two roads with a crossing, we study the relation between the average flow and the density of vehicles for different ratios between the road sizes. Simulations

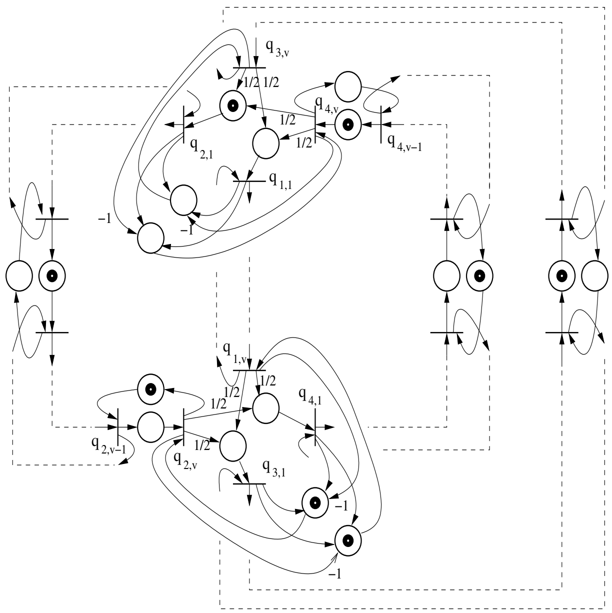

Fig. 5. The Petri net corresponding to four roads with two crossings.

show that for a fixed size of the system, the fundamental traffic diagram depends only on the ratio between the sum of sizes of the priority roads and the sum of sizes of the nonpriority roads. If we fix this ratio and we vary the sizes of the roads, we obtain, practically, the same diagram as shown on Figure 6. We show on Figure 7 the dependence of the fundamental diagram on this ratio.

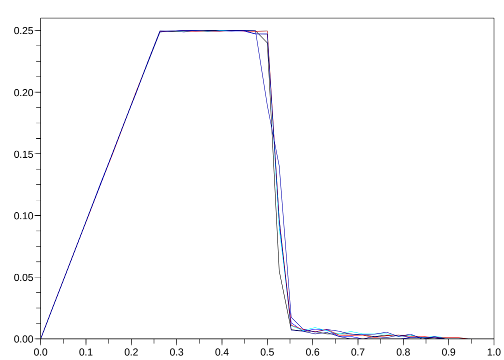

Fig. 6. Dependence of the global diagram on the sizes of four road without varying the ratio between the sum of sizes of priority roads and the sum of sizes of nonpriority roads.

We remark, first, that for reasonable ratios between these two sizes (one size is not negligible with respect to the other), the maximum flow is equal to half of the maximum flow obtained in the case of one circular road.

We remark, second, that a system of four roads with two crossings behaves like a system of two roads with one crossing, where the size of the priority road is the sum of sizes of two priority roads, and the size of the nonpriority road is the sum of sizes of two nonpriority roads. To

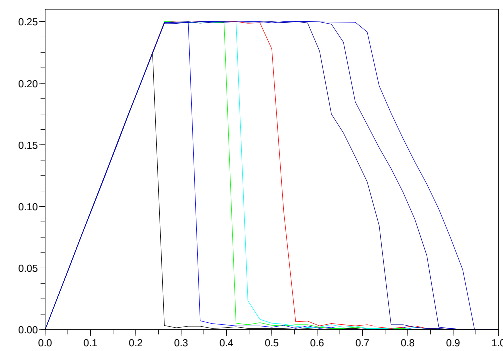

Fig. 7. Dependence of the global diagram on the ratio between the sum of the sizes of the priority roads and the sum of the sizes of the nonpriority roads.

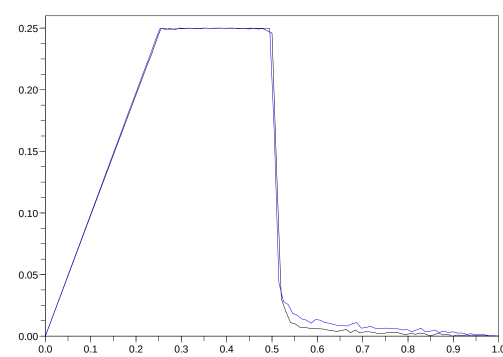

Fig. 8. Fundamental diagram for two roads (20 sections each one) with a crossing, and fundamental diagram for four roads (10 sections each one) with two crossings.

check this assumption, we consider two systems: – the first composed by two roads and one crossing, the size of each road being equal to 20 sections, – the second composed by four roads and two crossings, the size of each road being equal to 10 sections. In Figure 8 we compare the two diagrams.

On Table I, we show initial configurations and asymptotic stationary regimes corresponding to each phase of the fundamental diagram. The obtained three phases are qualitatively similar to those corresponding to a system of two roads with one crossing given in [2].

1) Low density phase : a periodic mode appears where vehicles circulate without being in the way of each other.
2) Average density phase : a periodic mode appears where the crossings are used at the maximum speed. Vehicles on the priority roads are not in the way of each other.

Vehicles on the nonpriority roads have to wait at the crossings.

3) High density phase : a deadlock appears, the nonpriority roads are full and two vehicles in the crossings want use these full roads.

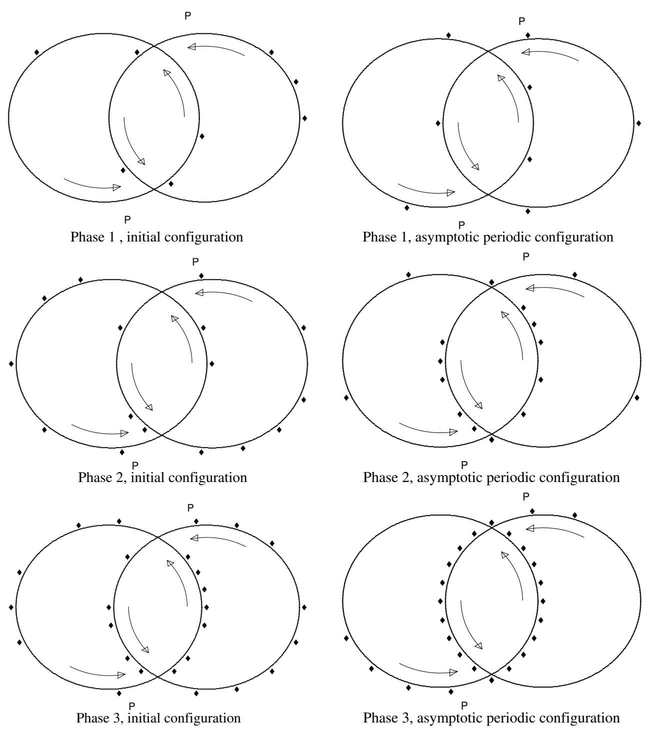

TABLE I

INITIAL AND ASYMPTOTIC (PERIODIC) CONFIGURATIONS FOR THE THREE PHASES.

C. Fundamental diagram for each road

In Table II we show the fundamental diagram corresponding to each of the four roads made of 20 sections. These diagrams are obtained by computing the average flow on each road (represented on the y-axis) for a discretization of the density set (x-axis). We can see that the two priority roads have practically the same diagram, idem for the nonpriority roads.

In Table III we show the aggregated fundamental diagram corresponding to the two priority roads and the aggregated fundamental diagram corresponding to the two nonpriority roads.

We can see that aggregated nonpriority roads have a fundamental diagram similar to the one of a circular road with a speed reducer [2], and that aggregated priority roads have a diagram with histeresis on which to one density does not correspond a unique flow. On this last diagram we see that the average density phase is reduced to the unique point of maximal flow. Added vehicles in this phase increase the density in the nonpriority roads but not in the priority roads.

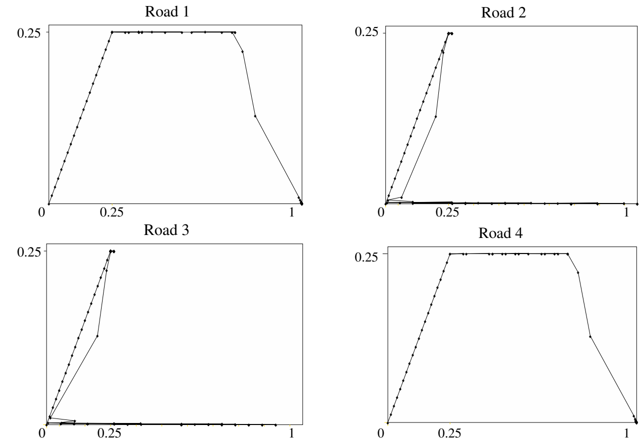

TABLE II

FUNDAMENTAL DIAGRAM OF EACH ROAD

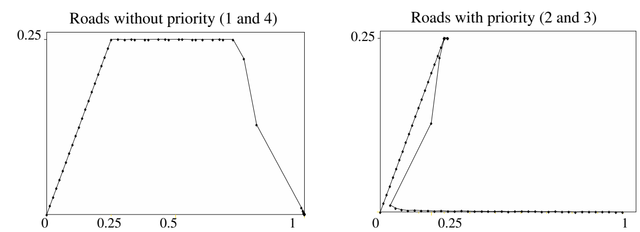

TABLE III

FUNDAMENTAL DIAGRAM OF PRIORITY ROADS AND FUNDAMENTAL DIAGRAM OF NONPRIORITY ROADS.

The global flow is constant during this phase and is at the maximum value.

D. The high density phase

In high enough density case, nonpriority roads fill up until one full circuit appears on the nonpriority roads. Then this crowded circuit creates a deadlock even if there is still a lot of places in the system see Table IV. This deadlock appears as soon as there is enough vehicles to full a circuit of non priority road. The worst appears surely.

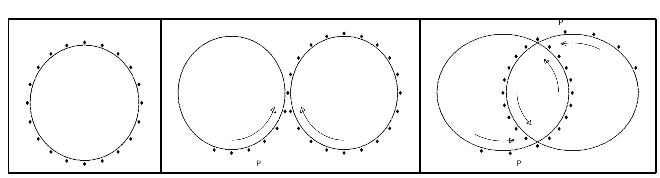

TABLE IV

LEFT: DEADLOCK I: 1 CIRCULAR ROAD. **MIDDLE:** DEADLOCK II: 2 ROADS WITH 1 CROSSING. **RIGHT:** DEADLOCK III: 4 ROADS WITH 2 CROSSINGS.

The qualitative results discussed here on the traffic phases for small systems are still valid for large network of roads. In a future paper we will explain a system methodology to build large network of roads and show that we obtain same kind of phase diagrams for large networks.

VI. Control and design of crossings.

To improve the traffic flow, we can use open loop or feedback traffic lights in order to avoid the blocking during the high density phase, and/or improve the crossings design in order to avoid the bottleneck at average density.

A. *Comparison of intersection policies*

We compare, on a symmetrical system (each road is composed of 25 sections), three crossing control policies :

- Right priority.
- Openloop traffic lights (the length of the light phases does not depend on the traffic).
- Feedback traffic lights control (the lengths of the phases depend on the vehicle number in the controlled streets (local feedback) or more generally on all the vehicle numbers of the different streets (for example determined by a LQG method). The experiments shown here have been done only with simple local feedback. With this simple local feedback we are able to reach the maximal flow possible (corresponding to a circular road with a retarder).

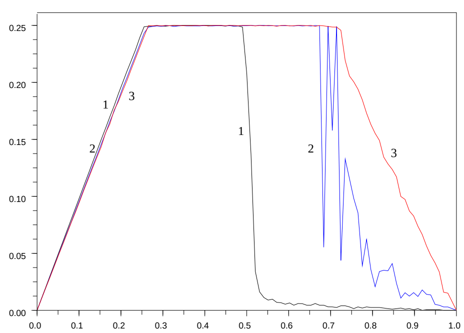

Fig. 9. Comparison of three management policies: – right priority (1), – open loop light control (2), – feedback light control (3).

On Figure 9, we give the diagrams corresponding to the three policies. We can see that traffic lights improve the flow only at medium and large density and does not reduce the flow at small density. Moreover, feedback policy seems to improve a lot at high density. The instability of the openloop policy at high density is still not clarified.

Another advantage of setting the feedback traffic lights is that they dissolve more quickly the jams than an open loop traffic light.

B. *Improvement of the crossing design.*

Here we study the influence of the buffer size in the crossing. In the previous model only one car can stay in the crossing (buffer size one). Let us study the case of a buffer size of two vehicles. The only modification to do in the Petri net modelling of Figure 1 is to change the constraint $a_\nu + a_{2\nu} + \bar{a}_\nu = 1$ into $a_\nu + a_{2\nu} + \bar{a}_\nu = 2$.

As shown on Figure 10, during the low density phase, the new design doesn't improve the flow. However, during the average density phase, the improvement is very important, and the maximum value of the flow reaches the maximum reached by one circular road without crossing !

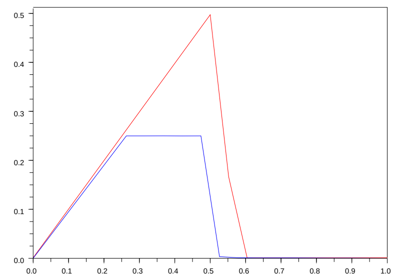

Fig. 10. Fundamental diagrams with buffer of size 1 and 2 at crossing.

During the high density phase, as soon as the number of vehicles exceeds the size of the nonpriority road the deadlock can appear. In this case with a feedback light control we can hope to reach the same diagram as in the unique circular road case.

VII. Complexity

The simulation of this model cost the integration of $X_{k+1} = (X_k H) \otimes D$ which is linear in the number of sections since the matrices H and D are sparse with a linear number of non null entries. To compute the average flow we have to wait the periodic regime. The Figure 11 shows simulation time to reach this mode, according to the size of the system. We see, on this example, that the time to reach the periodic regime increases linearly with the size of the system. We can conclude empirically that the cost is approximatively quadratic in the size of the network.

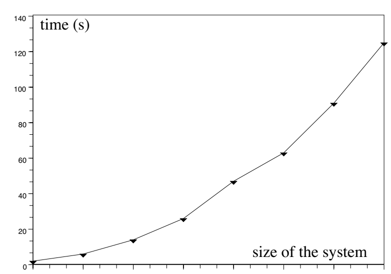

Fig. 11. The complexity of the simulation.

VIII. Conclusion

The fundamental traffic diagram of a system of four roads with two crossings have the same phases as the diagram of two roads with one crossing one. When the density is low, vehicles move freely without being in the way of each other. For medium densities vehicles on priority roads move freely when the other ones have to wait at crossings. The high density phase starts as soon the vehicles are enough to form a circuit of full nonpriority roads. For these densities there are deadlocks even if there is still a lot of places in the system. Light control avoid these deadlocks. With feedback light control we can reach almost the best possible diagram corresponding to a unique circular road with speed reducer. Moreover by buffering the crossing it is possible to reach the diagram of a unique circular road without crossing and without speed reducer. The analysis is clear numerically and need more work to be proven analytically using dynamic programming method or minplus algebra techniques.

APPENDIX

A. Resolution of a small system

Let us consider the eigenvalue problem associated to the simplest case $\nu = 2$ of the dynamics $(2), \cdots, (6)$ with the fluid approximation. Denoting λ the eigenvalue and X the corresponding eigen vector, we have to solve the system.

$$X^2\lambda = \bar{a}_2 X^1 X^3 / X^4 \oplus a_1 X^1 \;, \qquad (7)$$
$$X^4\lambda = \bar{a}_4 X^1 X^3 / (X^2\lambda) \oplus a_3 X^3 \;, \qquad (8)$$
$$X^1\lambda = a_2 \sqrt{X^2 X^4} \oplus \bar{a}_1 X^2 \;, \qquad (9)$$
$$X^3\lambda = a_4 \sqrt{X^2 X^4} \oplus \bar{a}^3 X^4 \;. \qquad (10)$$

Theorem 7: If $\bar{a}_2 \geq \bar{a}_4$, then there is a unique positive eigenvalue to $(7), \cdots, (10)$ which is the unique λ solution in X and λ of

$$X^2\lambda = a_1 X^1 \;, \qquad (11)$$
$$X^4\lambda = (\bar{a}_4 / a_1 \oplus a_3) X^3 \;, \qquad (12)$$
$$X^1\lambda = a_2 \sqrt{X^2 X^4} \oplus \bar{a}_1 X^2 \;, \qquad (13)$$
$$X^3\lambda = a_4 \sqrt{X^2 X^4} \oplus \bar{a}^3 X^4 \;. \qquad (14)$$

Proof: Multiplying (7) by X^4 and (8) by X^2 we obtain :

$$X^2 X^4 \lambda = \bar{a}_2 X^1 X^3 \oplus a_1 X^1 X^4 \;, \qquad (15)$$
$$X^2 X^4 \lambda = \bar{a}_4 X^1 X^3 / \lambda \oplus a_3 X^3 X^2 \;. \qquad (16)$$

But if $\lambda > 0$ then $\bar{a}_2 X^1 X^3 > \bar{a}_4 X^1 X^3 / \lambda$ and in the righthand of (15) the minimum is reached at $a_1 X^1 X^4$ because (16) shows that $X^2 X^4 \lambda \leq \bar{a}_4 X^1 X^3 / \lambda$, therefore (11) is true, and substituting $X^2\lambda$ in (8) we obtain (12).

But $(11), \cdots, (14)$ defines the troughput of an undiscounted Petri net and it follows that λ is unique and computable by dynamic programming techniques.

Conversely starting from $(11), \cdots, (14)$, knowing that $\lambda > 0$ by simple algebraic manipulation we show that λ is solution of $(7), \cdots, (10)$. ■

B. Simulation of an example.

Let's simulate the example given above with : $(a_1, \bar{a}_1) = (1, 0)$, $(a_2, a_4, \bar{a}_2, \bar{a}_4) = (0, 0, 1, 1)$, and $(a_3, \bar{a}_3) = (0, 1)$, and with $X_0 = 0$. This system can be written in the standard algebra as follows :

$$\begin{cases} X^2_{k+1} &= \min\{1 + X^1_k, 1 + X^1_k + X^3_k - X^4_k\}, \\ X^4_{k+1} &= \min\{X^3_k, 1 + X^1_k + X^3_k - X^2_{k+1}\}, \\ X^1_{k+1} &= \min\{X^2_k, \frac{1}{2}X^2_k + \frac{1}{2}X^4_k\}, \\ X^3_{k+1} &= \min\{1 + X^4_k, \frac{1}{2}X^2_k + \frac{1}{2}X^4_k\}. \end{cases}$$

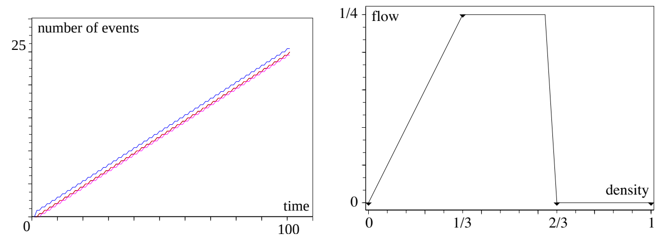

Fig. 12. **Left.** The number of events of X^1, X^2, X^3 and X^4. **Right.** The fundamental traffic diagram.

This situation corresponds to one vehicle in the system of three sections, that is $d = 1/3$. The obtained flow is $f = 1/4$ (Figure 12) which is the maximum flow. The diagram has been given also for fractional number of vehicles. The only other integer possibility is the case of 2 cars which gives a flow of 0.

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