BELLMAN PROCESSES

MARIANNE AKIAN, JEAN-PIERRE QUADRAT, AND MICHEL VIOT

1. INTRODUCTION

The mapping $\lim_{\varepsilon\to 0}\log_\varepsilon$ defines a morphism of algebra between the asymptotics (around zero) of positive real functions of a real number and the real numbers endowed with the two operations min and plus, indeed:

$$\lim_{\varepsilon\to 0}\log_\varepsilon(\varepsilon^a+\varepsilon^b)=\min(a,b),\quad \log_\varepsilon(\varepsilon^a\varepsilon^b)=a+b.$$

This morphism sends probability calculus into optimal control problems. Therefore almost of the concepts introduced in probability calculus have an optimization counterpart. The purpose of this paper is to make a presentation of known and new results of optimal control with this morphism in mind. The emphasis of this talk is $i)$ on the trajectory point of view by opposition to the cost point of view and $ii)$ on the optimization counterpart of processes with independent increments.

2. INF-CONVOLUTION AND CRAMER TRANSFORM

Definition 1. *Given two mappings f and g from $\mathbb{R}$ into $\overline{\mathbb{R}}\stackrel{\text{def}}{=}\mathbb{R}\cup\{+\infty\}$, the* inf-convolution of f and g *is the mapping $z\in\mathbb{R}\mapsto \inf_{x,y}\,[f(x)+g(y)\mid x+y=z]$. It is denoted $f\square g$. When f and g are lower bounded $f\square g$ is also lower bounded.*

Example 2. *For $m\in\mathbb{R}$ let us define the convex Dirac function:*

$$\delta_m^c(x)=\begin{cases}+\infty & \text{for } x\neq m,\\ 0 & \text{for } x=m,\end{cases}$$

and consider the function $\mathcal{M}^p_{m,\sigma}(x)=\frac{1}{p}(|x-m|/\sigma)^p$ for $p\geq 1$ with $\mathcal{M}^p_{m,0}=\delta^c_m$. We have the formula

$$\mathcal{M}^p_{m,\sigma}\square\mathcal{M}^p_{\bar m,\bar\sigma}=\mathcal{M}^p_{m+\bar m,[\sigma^{p'}+\bar\sigma^{p'}]^{1/p'}}\ \text{with } 1/p+1/p'=1\ .$$

This result is the analogue of

$$\mathcal{N}_{m,\sigma}*\mathcal{N}_{\bar m,\bar\sigma}=\mathcal{N}_{m+\bar m,\sqrt{\sigma^2+\bar\sigma^2}}$$

in the particular case $p=2$, where $\mathcal{N}_{m,\sigma}$ denotes the Gaussian law of mean m and standard deviation σ and $*$ the convolution operator.

Therefore there exists a morphism between the set of quadratic forms endowed with the inf-convolution operator and the set of exponentials of quadratic forms endowed with the convolution operator. This morphism is a particular case of the Cramer transform that we will define later. Let us first recall the definition of the Fenchel transform.

Definition 3. *Let $c \in \mathcal{C}_x$, where $\mathcal{C}_x$ denotes the set of mappings from $\mathbb{R}$ into $\overline{\mathbb{R}}$ convex, l.s.c. and proper. Its Fenchel transform is the function from $\mathbb{R}$ into $\overline{\mathbb{R}}$ defined by $\hat{c}(\theta) \stackrel{\text{def}}{=} [\mathcal{F}(c)](\theta) \stackrel{\text{def}}{=} \sup_x[\theta x - c(x)]$.*

Example 4. *The Fenchel transform of $\mathcal{M}^p_{m,\sigma}$ is*

$$[\mathcal{F}(\mathcal{M}^p_{m,\sigma})](\theta) = \frac{1}{p'}|\theta\sigma|^{p'} + m\theta \ ,$$

with $1/p + 1/p' = 1$. The particular case $p = 2$ corresponds to the characteristic function of a Gaussian law.

Theorem 5. *For $f, g \in \mathcal{C}_x$ we have $i)$ $\mathcal{F}(f) \in \mathcal{C}_x$, $ii)$ $\mathcal{F}$ is an involution that is $\mathcal{F}(\mathcal{F}(f)) = f$, $iii)$ $\mathcal{F}(f \square g) = \mathcal{F}(f) + \mathcal{F}(g)$, $iv)$ $\mathcal{F}(f + g) = \mathcal{F}(f) \square \mathcal{F}(g)$.*

Definition 6. *The Cramer transform $\mathcal{C}$ is a function from $\mathcal{M}$, the set of positive measures, into $\mathcal{C}_x$ defined by $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{F} \circ \log \circ \mathcal{L}$, where $\mathcal{L}$ denotes the Laplace transform.*

From the definition and the properties of the Laplace and Fenchel transform the following result is clear.

Theorem 7. *For $\mu, \nu \in \mathcal{M}$ we have $\mathcal{C}(\mu * \nu) = \mathcal{C}(\mu) \square \mathcal{C}(\nu)$.*

The Cramer transform changes the convolutions into inf-convolutions. In Table 1 we summarize the main properties and examples concerning the Cramer transform. The difficult results of this table can be found in Azencott [4]. In this table we have denoted by $\breve{A}$ the interior of the set A.

3. Decision Variables

The morphism between convolution and inf-convolution described in the previous section suggests the existence of a formalism adapted to optimization analogous to probability calculus. Some of the notions given here have been introduced in Bellalouna [5]. Another similar and independent work can be found in Del Moral [8]. We start by defining cost measures which can be seen as normalized idempotent measures of Maslov [13].

Definition 8. *We call* decision space *the triplet $(U, \mathcal{U}, \mathbb{K})$ where U is a topological space, $\mathcal{U}$ the set of the open sets of U and $\mathbb{K}$ a mapping from $\mathcal{U}$ into $\overline{\mathbb{R}}^+$ such that: $i)$ $\mathbb{K}(U) = 0$, $ii)$ $\mathbb{K}(\emptyset) = +\infty$, $iii)$ $\mathbb{K}\left(\bigcup_n A_n\right) = \inf_n \mathbb{K}(A_n)$ for any $A_n \in \mathcal{U}$.*

The mapping $\mathbb{K}$ is called a cost measure.

A map $c : u \in U \mapsto c(u) \in \overline{\mathbb{R}}^+$ such that $\mathbb{K}(A) = \inf_{u \in A} c(u),\ \forall A \subset U$ is called a cost density *of the cost measure $\mathbb{K}$.*

The set $D_c \stackrel{\text{def}}{=} \{u \in U \mid c(u) \neq +\infty\}$ is called the domain *of c.*

TABLE 1. Properties of the Cramer transform.

$\mathcal{M}$	$\log(\mathcal{L}(\mathcal{M})) = \mathcal{F}(\mathcal{C}(\mathcal{M}))$	$\mathcal{C}(\mathcal{M})$
μ	$\hat{c}_\mu(\theta) = \log \int e^{\theta x} d\mu(x)$	$c_\mu(x) = \sup_\theta(\theta x - \hat{c}(\theta))$
0	$-\infty$	$+\infty$
δ	0	δ^c
δ_a	θa	δ_a^c
$e^{-H(x)}$ $H(x) \stackrel{\text{def}}{=} 0$ if $x \geq O$ $+\infty$ elsewhere	$H(-\theta) - \log(-\theta)$	$H(x) - 1 - \log(x)$
$\lambda e^{-\lambda x - H(x)}$	$H(\lambda - \theta) + \log(\lambda/(\lambda - \theta))$	$H(x) + \lambda x - 1 - \log(\lambda x)$
$p\delta + (1-p)\delta_1$	$\log(p + (1-p)e^\theta)$	$x \log(\frac{x}{1-p})$ $+(1-x)\log(\frac{1-x}{p})$ $+H(x) + H(1-x)$
$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-m)^2/\sigma^2}$	$m\theta + \frac{1}{2}(\sigma\theta)^2$	$\mathcal{M}^2_{m,\sigma}$
Inf. divis. distrib. Feller [10]	$m\theta + \frac{1}{p'}\lvert\sigma\theta\rvert^{p'}$	$\mathcal{M}^p_{m,\sigma}$ with $p > 1,\ 1/p + 1/p' = 1$
	$m\theta + H(-\theta + 1/\sigma)$ $+H(\theta + 1/\sigma)$	$\mathcal{M}^1_{m,\sigma}$
	$a\theta \vee b\theta,\ a \leq b$	$H(x-a) + H(-x+b)$
$\mu * \nu$	$\hat{c}_\mu + \hat{c}_\nu$	$c_\mu \square c_\nu$
$k\mu$	$\log(k) + \hat{c}$	$c - \log(k)$
$\mu \geq 0$	$\hat{c}$ convex l.s.c.	c convex l.s.c.
$m_0 \stackrel{\text{def}}{=} \int \mu$	$\hat{c}(0) = \log(m_0)$	$\inf_x c(x) = -\log(m_0)$
$m_0 = 1$	$\hat{c}(0) = 0$	$\inf_x c(x) = 0$
$\mathcal{P} \stackrel{\text{def}}{=} \{\mu \geq 0 \mid m_0 = 1\}$ $S_\mu \stackrel{\text{def}}{=} \overline{\text{cvx}(\text{supp}(\mu))}$ $\mu \in \mathcal{P}$	$\hat{c}$ strictly convex in $D_{\hat{c}}$ C^∞ in $\breve{D}_{\hat{c}}$	$D_c \stackrel{\text{def}}{=} \text{dom}(c)$ $\breve{D}_c = \breve{S}_\mu$ C^1 in $\breve{D}_c$
$m_0 = 1,\ m \stackrel{\text{def}}{=} \int x\mu$	$\hat{c}'(0) = m$	$c(m) = 0$
$m_0 = 1,\ m_2 \stackrel{\text{def}}{=} \int x^2\mu$	$\hat{c}''(0) = \sigma^2 \stackrel{\text{def}}{=} m_2 - m^2$	$c''(m) = 1/\sigma^2$
$m_0 = 1$ $\hat{c} = \lvert\sigma\theta\rvert^{p'}/p' + o(\lvert\theta\rvert^{p'})$	$\hat{c}^{(p')}(0^+) = \Gamma(p')\sigma^{p'}$	$c^{(p)}(0^+) = \Gamma(p)/\sigma^p$

Theorem 9. *Given a l.s.c. positive real valued function c such that $\inf_u c(u) = 0$, $\mathbb{K}(A) = \inf_{u \in A} c(u)$ for all A open set of U defines a cost measure. Conversely any cost measure defined on the open sets of a Polish space admits a unique minimal extension $\mathbb{K}_*$ to $\mathcal{P}(U)$ (the set of the parts of U) having a density c which is a l.s.c. function on U satisfying $\inf_u c(u) = 0$.*

Proof. This precise result is proved in Akian[1]. See Maslov[13] for the first result of this kind. See also Del Moral[8] for analogous results. □

We have seen that the images by the Cramer transform of the probability measures are C^1 and convex cost density functions.

By analogy with the conditional probability we define now the conditional cost excess.

Definition 10. *The* conditional cost excess *to take the best decision in A knowing that it must be taken in B is*

$$\mathbb{K}(A|B) \stackrel{\text{def}}{=} \mathbb{K}(A \cap B) - \mathbb{K}(B) \ .$$

Definition 11. *By analogy with random variables we define decision variables and related notions.*

1. *A* numerical decision vector X *on* $(U, \mathcal{U}, \mathbb{K})$ *is a mapping from* U *into* $\mathbb{R}^n$. *It induces* $\mathbb{K}_X$ *a cost measure on* $(\mathbb{R}^n, \mathcal{B})$ *($\mathcal{B}$ denotes the set of open sets of* $\mathbb{R}^n$*) defined by* $\mathbb{K}_X(A) = \mathbb{K}_*(X^{-1}(A)),\ \forall A \in \mathcal{B}$. *The cost measure* $\mathbb{K}_X$ *has a l.s.c. density denoted* c_X . *When the vector is of dimension 1 we call it a* decision variable.
2. *A decision variable is said* regular *when its cost measure is regular.*
3. *Two decision variables* X *and* Y *are said* independent *when:*
$$c_{X,Y}(x, y) = c_X(x) + c_Y(y).$$
4. *The* conditional cost excess *of* X *knowing* Y *is defined by:*
$$c_{X|Y}(x, y) \overset{\text{def}}{=} \mathbb{K}_*(X = x \mid Y = y) = c_{X,Y}(x, y) - c_Y(y).$$
5. *The* optimum *of a decision variable is defined by*
$$\mathbb{O}(X) \overset{\text{def}}{=} \arg\min_x c_X(x)$$
when the minimum exists. When a decision variable X *satisfies* $\mathbb{O}(X) = 0$ *we say that it is* centered.
6. *When the optimum of a decision variable* X *is unique and when near the optimum, we have:*
$$c_X(x) = \frac{1}{p}\left|\frac{x - \mathbb{O}(x)}{\sigma}\right|^p + o(|x - \mathbb{O}(x)|^p)\ ,$$
we define the sensitivity of order p *of* $\mathbb{K}$ *by* $\mathbb{S}^p(X) \overset{\text{def}}{=} \sigma$. *When a decision variable satisfies* $\mathbb{S}^p(X) = 1$ *we say that it is* of order p and normalized. *When we speak of sensitivity without making the order precise, we implicitly mean that this order is 2.*
7. *The numbers*
$$|X|_p \overset{\text{def}}{=} \inf\left\{\sigma \mid c_X(x) \geq \frac{1}{p}|(x - \mathbb{O}(X))/\sigma|^p\right\} \text{ and } \|X\|_p \overset{\text{def}}{=} |X|_p + |\mathbb{O}(X)|$$
define respectively a seminorm and a norm on the set of decision variables having a unique optimum such that $\|X\|_p$ *is bounded. The corresponding set of decision variables is called* $\mathbb{D}^p$.
8. *The* mean *of a decision variable* X *is* $\mathbb{M}(X) \overset{\text{def}}{=} \inf_x(x + c_X(x))$, the conditional mean *is* $\mathbb{M}(X \mid Y = y) \overset{\text{def}}{=} \inf_x(x + c_{X|Y}(x, y))$.
9. *The* characteristic function *of a decision variable is* $\mathbb{F}(X) \overset{\text{def}}{=} \mathcal{F}(c_X)$ *(clearly* $\mathbb{F}$ *characterizes only decision variables with cost in* $\mathcal{C}_x$*).*

Example 12. *For a decision variable* X *of cost* $\mathcal{M}^p_{m,\sigma}$, $p > 1$, *we have*
$$\mathbb{O}(X) = m,\ \mathbb{S}^p(X) = |X|_p = \sigma,\ \mathbb{M}(X) = m - \frac{1}{p'}\sigma^{p'}\ .$$

The role of the Laplace or Fourier transform in the probability calculus is played by the Fenchel transform in the decision calculus.

Theorem 13. *If the cost density of a decision variable is convex, admits a unique minimum and is of order p, we have:*

$$\mathbb{F}(X)'(0) = \mathbb{O}(X), \quad [\mathbb{F}(X - \mathbb{O}(X))]^{(p)}(0) = \Gamma(p')[\mathbb{S}^p(X)]^{p'} .$$

Theorem 14. *For two independent decision variables X and Y of order p and $k \in \mathbb{R}$ we have*

$$c_{X+Y} = c_X \Box c_Y, \quad \mathbb{F}(X+Y) = \mathbb{F}(X) + \mathbb{F}(Y), \quad [\mathbb{F}(kX)](\theta) = [\mathbb{F}(X)](k\theta) ,$$

$$\mathbb{O}(X+Y) = \mathbb{O}(X) + \mathbb{O}(Y), \quad \mathbb{O}(kX) = k\mathbb{O}(X), \quad \mathbb{S}^p(kX) = |k|\mathbb{S}^p(X) ,$$

$$[\mathbb{S}^p(X+Y)]^{p'} = [\mathbb{S}^p(X)]^{p'} + [\mathbb{S}^p(Y)]^{p'}, \quad (|X+Y|_p)^{p'} \leq (|X|_p)^{p'} + (|Y|_p)^{p'} .$$

4. Independent Sequences of Decision Variables

We consider, in this section, sequences of independent decision variables and the analogues of the classical limit theorems of the probability calculus.

Definition 15. *A* sequence of independent decision variables identically cost of cost *c on $(U, \mathcal{U}, \mathbb{K})$ (i.i.c.) is an application X from U into $\mathbb{R}^{\mathbb{N}}$ which induces a density cost satisfying*

$$c_X(x) = \sum_{i=0}^{\infty} c(x_i), \quad \forall x = (x_0, x_1, \dots) \in \mathbb{R}^{\mathbb{N}} .$$

Remark 16.
1. *The cost density is finite only on minimizing sequences of c, elsewhere it is equal to $+\infty$.*
2. *We have defined a decision sequence by its density and not by its value on the open sets of $\mathbb{R}^{\mathbb{N}}$ because the density can be defined easily.*

In order to state the limit theorems, we define several type of convergence of sequences of decision variables.

Definition 17. *For the numerical decision sequence $\{X_n, n \in \mathbb{N}\}$ we say that*

1. $X_n$converges weakly *towards X, denoted $X_n \overset{w}{\to} X$, if for all f in $\mathcal{C}_b(\mathbb{R})$ (where $\mathcal{C}_b(\mathbb{R})$ denotes the set of uniformly continuous and lower bounded functions on $\mathbb{R}$) we have* $\lim_n \mathbb{M}[f(X_n)] = \mathbb{M}[f(X)]$ *. When the test functions used are the set of affine functions we say that it converges* weakly$*$ *($w*$);*
2. $X_n \in \mathbb{D}^p$ converges in p-sensitivity *towards $X \in \mathbb{D}^p$ denoted $X_n \xrightarrow{\mathbb{D}^p} X$, if* $\lim_n \|X_n - X\|_p = 0$;
3. X_n converges in cost *towards X, denoted $X_n \xrightarrow{\mathbb{K}} X$, if for all $\epsilon > 0$ we have* $\lim_n \mathbb{K}\{u \mid |X_n(u) - X(u)| \geq \epsilon\} = +\infty$;
4. X_n converges almost surely *towards X, denoted $X_n \xrightarrow{a.s.} X$, if we have* $\mathbb{K}\{u \mid \lim_n X_n(u) \neq X(u)\} = +\infty$.

Some relations between these different kinds of convergence are given in the following theorem.

Theorem 18. 1. *Convergence in sensitivity implies convergence in cost but the converse is false.*
2. *Convergence in cost implies almost sure convergence and the converse is false.*
3. *Almost sure convergence does not imply weak convergence.*
4. *Convergence in cost implies the weak convergence.*

Proof. See Akian[2]. In Bellalouna[5] the point 3 has been proved previously. ☐

We have the analogue of the law of large numbers.

Theorem 19. *Given a sequence of independent decision variables belonging to $\mathbb{D}^p,\ p \geq 1$, identically cost (i.i.c.) $\{X_n,\ n \in \mathbb{N}\}$ we have:*

$$\lim_{N\to\infty} Y_N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=0}^{N-1} X_n = \mathbb{O}(X_0)\ ,$$

where the limit can be taken in the sense of the weak, almost sure, cost and p-sensitivity convergence.

Proof. We have only to estimate the convergence in sensitivity. The results follows from simple computation of the p-seminorm of Y_N. It satisfies $(|Y_N|_p)^{p'} = N(|X_0|_p)^{p'}/N^{p'}$ thanks to theorem 14. ☐

We have the analogue of the central limit theorem of the probability calculus.

Theorem 20. *Given an i.i.c. sequence $\{X_n, n \in \mathbb{N}\}$ centered of order p we have*

$$\text{weak} * \lim_N\ Z_N \stackrel{\text{def}}{=} \frac{1}{N^{1/p'}}\sum_{n=0}^{N-1} X_n = X\ ,$$

where X is a decision variable with cost equal to $\mathcal{M}^p_{0,\mathbb{S}^p(X_0)}$.

Proof. We have $\lim_N [\mathbb{F}(Z_N)](\theta) = \frac{1}{p'}[\mathbb{S}^p(X_0)\theta]^{p'}$. ☐

5. Bellman Chains

We can generalize i.i.c. sequences to the analogue of Markov chains that we will call Bellman chains.

Definition 21. *A finite valued Bellman chain (E, C, ϕ) with $i)$ E a finite set called the state space of $|E|$ elements, $ii)$ $C : E \times E \mapsto \overline{\mathbb{R}}$ satisfying* $\inf_y C_{xy} = 0$ *called the transition cost, $iii)$ ϕ is a cost measure on E called the initial cost, is the decision sequence $\{X_n\}$ on $(U, \mathcal{U}, \mathbb{K})$, taking its values in E, such that $c_X(x \stackrel{\text{def}}{=} (x_0, x_1, \dots)) = \phi_{x_0} + \sum_{i=0}^{\infty} C_{x_i x_{i+1}}\ ,\ \forall x \in E^{\mathbb{N}}$.*

Theorem 22. *For any function f from E into $\overline{\mathbb{R}}$, a Bellman chain satisfies the Markov property $\mathbb{M}\{f(X_n) \mid X_0, \dots, X_{n-1}\} = \mathbb{M}\{f(X_n) \mid X_{n-1}\}$.*

The analogue of the forward Kolmogorov equation giving a way to compute recursively the marginal probability to be in a state at a given time is the following Bellman equation.

Theorem 23. *The marginal cost $w_x^n = \mathbb{K}(X_n = x)$ of a Bellman chain is given by the recursive forward equation:*

$$w^{n+1} = w^n \otimes C \stackrel{\text{def}}{=} \min_{x \in E}(w_x^n + C_{x.}) \ ,$$

with $w^0 = \phi$.

The cost measure of a Bellman chain is normalized which means that its infimum on all the trajectories is 0. In some applications we would like to avoid this restriction. This can be done by introducing the analogue of the multiplicative functionals of the trajectories of a stochastic process.

Theorem 24. *The value*

$$v_x^n \stackrel{\text{def}}{=} \mathbb{M}\{\sum_{k=n}^{N-1} f(X_k) + \psi(X_N) \mid X_n = x\}$$

with $f, \psi \in \overline{\mathbb{R}}^{|E|}$ can be computed recursively by

$$v^n = F \otimes C \otimes v^{n+1} = f(.) + \min_y (C_{.y} + v_y^{n+1}), \quad v^N = \psi \ ,$$

where F is the $(|E|, |E|)$ matrix defined by $F_{xy} \stackrel{\text{def}}{=} f_x$ if $x = y$ and $+\infty$ elsewhere. The matrix F can be seen as a normalizing factor of the transition cost.

6. Continuous-Time Bellman Processes

We can define easily continuous time decision processes which correspond to deterministic controlled processes with a cost associated to each trajectory. We discuss here only decision processes with continuous trajectories.

Definition 25. *Associated to continuous time decision processes we have the following definitions.*

1. *A* continuous time Bellman process *X_t on $(U, \mathcal{U}, \mathbb{K})$, with continuous trajectories, is a function from U into $\mathcal{C}(\mathbb{R}^+)$ (where $\mathcal{C}(\mathbb{R}^+)$ denotes the set of continuous functions over $\mathbb{R}^+$ into $\mathbb{R}$) having the cost density*

$$c_X(x(\cdot)) \stackrel{\text{def}}{=} \phi(x(0)) + \int_0^\infty c(t, x(t), x'(t))dt \ ,$$

   with $c(t, \cdot, \cdot)$ a family of transition costs (that is a mapping c from $\mathbb{R}^3$ into $\overline{\mathbb{R}}^+$ such that $\inf_y c(t, x, y) = 0, \ \forall t, x$) and ϕ a cost density on $\overline{\mathbb{R}}$. When the integral is not defined the cost is by definition equal to $+\infty$.
2. *The Bellman process is said* homogeneous if c does not depend on the time t.

3. *The Bellman process is said* with independent increments *if c does not depend on the state x. Moreover if this process is homogeneous c is reduced to the cost density of a decision variable.*

Example 26. *The following processes are fundamental.*

1. *The* p-Brownian decision process, *denoted by B_t^p, is the process with independent increments and transition cost density $c(t,x,y) = \frac{1}{p}y^p$.*
2. *The* p-diffusion decision process, *denoted by $\mathcal{M}^p_{m,\sigma,t}$ will correspond to the transition cost density $c(t,x,y) = \mathcal{M}^p_{m(t,x),\sigma(t,x)}(y)$.*

As in the discrete time case, the marginal cost to be in a state x at a time t can be computed recursively using a forward Bellman equation.

Theorem 27. *The marginal cost $w(t,x) \stackrel{\text{def}}{=} \mathbb{K}(X_t = x)$ is given by the Bellman equation:*

$$\partial_t w + \hat{c}(\partial_x w) = 0, \quad w(0,x) = \phi(x) \ , \tag{1}$$

where $\hat{c}$ means here $[\hat{c}(\partial_x w)](t,x) \stackrel{\text{def}}{=} \sup_y [y\partial_x w(t,x) - c(t,x,y)]$.

Example 28. *Let us give two examples where explicit computation can be made.*

1. *For the brownian decision process B_t^p starting from 0, the marginal cost to be at time t in the state x satisfies the Bellman equation:*
$$\partial_t w + (1/p')[\partial_x w]^{p'} = 0, \quad w(0,\cdot) = \delta^c \ .$$
Its solution can be computed explicitly, it is $w(t,x) = \mathcal{M}^p_{0,t^{1/p'}}(x)$. Therefore we have
$$\mathbb{M}[f(B_t^p)] = \inf_x \left[f(x) + \frac{x^p}{pt^{\frac{p}{p'}}} \right] .$$
2. *The marginal cost of the p-diffusion decision process $\mathcal{M}^p_{m,\sigma,t}$ starting from O, satisfies the Bellman equation: $\partial_t w + m\partial_x w + \frac{1}{p'}[\sigma\partial_x w]^{p'} = 0$, $w(0,\cdot) = \delta^c$. Explicit solution is known only in particular cases for instance when $m(t,x) = -\alpha x$ and σ is constant. This case gives a generalization of the well known LQ problem. The solution is $w(t,x) = (1/p)q(t)(x/\sigma)^p$ where $q(t)$ satisfies the Bernouilli equation*
$$\dot{q}/p - \alpha q + q^{p'}/p' = 0, \quad q(0) = +\infty \ .$$

The backward Bellman equation gives a mean to compute the analogue of the multiplicative functionals of a stochastic process.

Theorem 29. *The functional $v(t,x) = \mathbb{M}\{\psi(X_T) \mid X_t = x\}$, where ψ is a mapping from $\mathbb{R}$ into $\overline{\mathbb{R}}$ and X_t is a Bellman process of transition cost c and initial cost ϕ, can be computed recursively by the Bellman equation*

$$\partial_t v(t,x) + \inf_y [y\partial_x v(t,x) + c(t,x,y)] = 0, \quad v(T,x) = \psi(x) \ .$$

Moreover we have $\mathbb{M}[\psi(X_T)] = \inf_x\ [w(t,x) + v(t,x)],\ \forall t,$ *where* w *is the solution of (1).*

Example 30. *In the case of the p-Brownian decision process* $X_t = B_t^p$ *we have an explicit formula dual of the formula given in the previous example:*

$$v(t,x) \stackrel{\text{def}}{=} \mathbb{M}\{\psi(X_T) \mid X_t = x\} = \inf_y \left[\psi(y) + \frac{1}{p}(y-x)^p/(T-t)^{p/p'}\right] ,$$

moreover when the initial cost of the brownian is ϕ *we have:*

$$\mathbb{M}\{\psi(X_T)\} = \inf_{x,y} \left[\psi(y) + \frac{1}{p}(y-x)^p/T^{\,p/p'} + \phi(x)\right] .$$

Other explicit formulas involving stopping time, in the particular case p=2, are given in [15].

Theorem 31. *Given n an integer,* $h = T/n$*, the linear interpolation of the Bellman chain of transition cost* $c_h(x,y)$ *defines a decision process with trajectories in* $\mathcal{C}[0,T]$*. This process converges weakly, when n goes to* ∞*, towards the order p process* $\mathcal{M}^p_{m,\sigma,t}$ *as soon as:*

$$[\mathcal{F}(c_h(x, x+\cdot)](\theta) = [m(x)\theta + (1/p')(\sigma(x)\theta)^{p'}]h + o(h) ,$$

or equivalently if

$$i)\ \mathbb{O}_{X_k=x}[X_{k+1} - X_k] = m(x)h + o(h) ,$$

$$ii)\ \mathbb{S}^p_{X_k=x}[X_{k+1} - X_k] = \sigma(x)h^{1/p'} + o(h^{1/p'}) .$$

This result is called min-plus invariance principle. See Samborski and Dudnikov in these proceedings for related results.

7. Conclusion

Let us conclude by summarizing the morphism between probability calculus and decision calculus in Table 2.

Notes 32. *Bellman* [6] *was aware of the interest of the Fenchel transform (which he calls max transform) for the analytic study of the dynamic programming equations. The Cramer transform is an important tool in large deviations literature* [4],[12],[17]. *Maslov has developed a theory of idempotent integration* [13]. *In* [15] *and* [7] *the law of large numbers and the central limit theorem for decision variables has been given in the particular case* $p = 2$. *In two independent works* [5] *and* [8] *the study of decision variables have been started. Some aspects of* [18] *are strongly related to this morphism between probability and decision calculus in particular the morphism between LQG and LEG problem and the link with* H_∞ *problem. In* [14] *idempotent Sobolev spaces have been introduced as a way to study HJB equation as a linear object.*

We would also like to thank P.L. Lions and R. Azencott for some nice comments showing us the role of the Cramer transform in the morphism

TABLE 2. Morphism between probability calculus and decision calculus.

Probability	Decision
$+$	min
$\times$	$+$
Measure: $\mathbb{P}$	Cost: $\mathbb{K}$
$\int d\mathbb{P}(x) \stackrel{\text{def}}{=} 1$	$\min_x \mathbb{K}(x) \stackrel{\text{def}}{=} 0$
$\mathcal{N}_{m,\sigma}$	$\mathcal{M}^2_{m,\sigma}$
Convolution	Inf-Convolution
Laplace	Fenchel
Random Variable: X	Decision Variable: X
$\mathbb{E}(X) \stackrel{\text{def}}{=} \int x dF(x)$	$\mathbb{M}(X) \stackrel{\text{def}}{=} \inf_x \{x + \mathbb{K}(x)\}$
	$\mathbb{O}(X) \stackrel{\text{def}}{=} \arg\min_x \mathbb{K}(x)$
$\sigma(X) \stackrel{\text{def}}{=} \sqrt{\mathbb{E}[(X - \mathbb{E}(X))^2]}$	$\mathbb{S}^2(X) \stackrel{\text{def}}{=} \sqrt{1/c''_X(\mathbb{O}(X))}$
$\phi(X) \stackrel{\text{def}}{=} \mathbb{E}(e^{\theta X})$	$\mathbb{F}(X) \stackrel{\text{def}}{=} -\mathbb{M}(-\theta X)$
$\log \phi(X)'(0) = \mathbb{E}(X)$	$\mathbb{F}(X)'(0) = \mathbb{O}(X)$
$\log \phi(X)''(0) = (\sigma(X))^2$	$\mathbb{F}(X)''(0) = (\mathbb{S}^2(X))^2$
Markov Chain	Bellman Chain
Kolmogorov Eq.	Bellman Eq.
Stochastic Process	Decision Process
Brownian Motion	2-Brownian Decision Process
Diffusion Process	2-Diffusion Decision Process
Heat Equation	Quadratic HJB Equation
$\partial_t + m(x)\partial_x + \frac{1}{p'}(\sigma(x))^{p'}\partial_x^{(p')}$	$\partial_t v + m(x)\partial_x v - \frac{1}{p'}\lvert\sigma(x)\partial_x v\rvert^{p'}$
Gaussian Kernel	Quadratic Kernel
$\frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$	$(x-y)^2/2t$
Invariance Principle	(Min,+) Invariance Principle

between probability and decision calculus. These comments have been an important step in the maturation process of this paper.

REFERENCES

[1] M. Akian : *Idempotent integration and cost measures,* INRIA Report (1994), to appear.
[2] M. Akian : *Theory of cost measures: convergence of decision variables,* INRIA Report (1994), to appear.
[3] H. Attouch , R.J.B. Wets : *Isometries of the Legendre-Fenchel transform,* Transactions of the American Mathematical Society 296 (1986) 33–60.
[4] R. Azencott, Y. Guivarc'h, R.F.Gundy, : *Ecole d'été de Saint Flour 8,* Lect. Notes in Math., Springer-Verlag, Berlin (1978).
[5] F. Bellalouna : *Processus de décision min-markovien,* Thesis dissertation, University of Paris-Dauphine (1992).

[6] R. Bellman, W. Karush : *Mathematical programming and the maximum transform,* SIAM Journal of Applied Mathematics 10 (1962).

[7] F. Baccelli, G. Cohen, G.J. Olsder, J.P.Quadrat, :*Synchronization and Linearity: an Algebra for Discrete Event Systems,* John Wiley and Sons, New York (1992).

[8] P. Del Moral : *Résolution particulaire des problèmes d'estimation et d'optimisation non-linéaires,* Thesis dissertation, Toulouse, France (1994).

[9] P. Del Moral, T. Thuillet, G. Rigal, G. Salut : *Optimal versus random processes: the non-linear case,* LAAS Report, Toulouse, France (1990).

[10] W. Feller : *An Introduction to Probability Theory and its Applications,* John Wiley and Sons, New York (1966).

[11] W. Fenchel: *On the conjugate convex functions,* Canadian Journal of Mathematics 1 (1949) 73–77.

[12] M.I. Freidlin, A.D. Wentzell : *Random Perturbations of Dynamical Systems,* Springer-Verlag, Berlin (1979).

[13] V. Maslov: *Méthodes Opératorielles,* Éditions MIR, Moscou (1987).

[14] V Maslov, S.N. Samborski : *Idempotent Analysis. Advances In Soviet Mathematics,* 13 Amer. Math. Soc., Providence, (1992).

[15] J.P. Quadrat : *Théorèmes asymptotiques en programmation dynamique,* Note CRAS Paris 311 (1990) 745–748.

[16] R.T. Rockafellar : *Convex Analysis,* Princeton University Press Princeton, N.J. (1970).

[17] S.R.S. Varadhan : *Large deviations and applications,* CBMS-NSF Regional Conference Series in Applied Mathematics N.46, SIAM, Philadelphia, Penn., (1984).

[18] P. Whittle : *Risk Sensitive Optimal Control,* John Wiley and Sons, New York (1990).

MARIANNE AKIAN, INRIA-ROCQUENCOURT, B.P. 105, 78153 LE CHESNAY CEDEX, FRANCE

JEAN-PIERRE QUADRAT, INRIA-ROCQUENCOURT, B.P. 105, 78153 LE CHESNAY CEDEX, FRANCE

MICHEL VIOT, CMA, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE