FROM FIRST TO SECOND-ORDER THEORY OF LINEAR DISCRETE EVENT SYSTEMS

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Abstract. For timed event graphs, linear models were obtained using dioid algebra. After describing backward equations which solve an optimal tracking problem and which introduce co-state variables, this paper presents preliminary results concerning the matrix of 'ratios' (i.e. conventional differences) of co-states over states: this matrix sounds like a Riccati matrix, although a neat analogue to a Riccati equation has not been found yet.

Keywords. Discrete event systems, timed event graphs, dioid algebra, residuation, co-state equations

1 INTRODUCTION

In the last ten years, a new paradigm has emerged under the now classical name of 'discrete event (dynamic) systems' (DEDS). Among the various aspects of these systems[1], the scope is here *performance evaluation*. Hence the focus is on dates of occurrence of events, number of events occurred within a certain time interval, etc. It was shown that when synchronization is the basic phenomenon which drives the dynamics of such systems (this supposes that such issues as concurrency, scheduling, etc., have been already handled by some predefined rules), it is possible to develop a theory which offers a striking analogy with conventional *linear* system theory. The clue here is the use of an algebraic framework particularly suited to handle synchronization phenomena (Baccelli et al., 1992), known under the name of dioid algebra (among other names). In the language of Petri net theory, the class of systems one can study with this approach is the class of (timed) event graphs.

There are indeed several points of view for modeling systems in this perspective and, according to the one which is adopted, different dioid algebras are used. One can mention the 'dater', the 'counter', and the 'two-dimensional domain' points of view. Using either of these modeling approaches, several notions have been introduced so far and a fair amount of results have been obtained, which parallel those of conventional system theory. Among others, one may quote 'state space' and input-output (transfer matrix) representations of systems, eigenvalue-eigenvector pairs and their connection with asymptotic periodic regimes, frequency responses, stabilization by feedback control, etc. (see (Baccelli et al., 1992) and the references therein). Most of these developments may be viewed as pertaining to the 'first-order' theory of such systems.

[1] roughly speaking, they can be classified into *qualitative* and *quantitative* aspects

Optimization problems regarding resources have also been addressed (Gaubert, 1992). Other quantities of interest, such as sojourn times of tokens in some parts of an event graph, or dually numbers of tokens in process in certain parts of the system, require considering the (conventional) differences of the variables which are handled in the first-order theory. It should be realized that (conventional) subtraction is a *nonlinear* operation in the dioid setting considered in this theory: more precisely, subtraction can be related to the so-called residual of $+$ (which plays the role of multiplication), hence it is a kind of inverse operation of multiplication (a sort of 'division'). Residuation is a well-established theory in the framework of lattice-ordered semigroups (which is another view point on dioids) and it proves to be very useful in dealing with all these issues.

From the system-theoretic point of view, it turns out that manipulating such nonlinear objects can be viewed as making the first steps into the realm of *second-order* system theory because of the analogy these quantities offer with correlations, autocorrelations or Riccati matrices. A first account of these new developments can be found in (Max Plus, 1991). In this reference, the focus was on sojourn times and numbers of tokens in process. Algebraically, these quantities behave as correlations.

There are other interesting quantities which are obtained by conventional differences. For example, in the dater point of view, forward dynamic equations provide the earliest possible dates at which successive events (transition firings) can occur once the 'inputs' (e.g. source transition firing times) are given. In this paper, it will be shown that some *backward* dynamic equations in a dual algebra can be derived in order to calculate the latest possible firing times of all transitions (especially the source transitions) once output (sink) transition firing times are given. This amounts to 'inverting' a dynamical system and it appeals again to the theory of residuation (Blyth and Janowitz, 1972, Baccelli et al., 1992, Cuninghame-Green, 1979). The backward equations are remindful of co-state (or adjoint-state) equations in optimal control theory. Once earliest and latest dates have been obtained by dynamic forward and backward equations, one may be interested in subtracting the former from the later to obtain 'margins' on processing times, which allows to answer questions such as: "How long a particular event can be delayed without altering the delivering time of products at the outlet of the system?". From the mathematical point of view, one is again dealing with quantities which are derived from other quantities by conventional subtraction (that is, by dioid division). In the present case, the ratio of a variable similar to a co-state variable by a variable similar to a state variable[2] was considered: this is again very reminiscent of Riccati matrices in LQ optimal control problems. This paper proposes preliminary developments of this second-order theory of discrete event systems.

Throughout this paper, and without further mention, the technical background which is only briefly alluded to because of the lack of space can be found in more details in (Baccelli et al., 1992).

2 DIOIDS

A *dioid* is a set endowed with two internal operations denoted $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted ε and e respectively, such that $\oplus$ is also commutative and idempotent (i.e. $a \oplus a = a$), $\otimes$ is distributive with respect to $\oplus$, and ε is absorbing for the product (i.e. $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a$). When $\otimes$ also is commutative, the dioid is said to be commutative. The symbol $\otimes$ is often omitted.

The set $\mathbb{Z} \cup \{-\infty\}$ with $\max$ as $\oplus$ and $+$ as $\otimes$ is a dioid of interest for us, and it is denoted $\mathbb{Z}_{\max}$. Another dioid that will be used is $\mathbb{Z}_{\min} = (\mathbb{Z} \cup \{+\infty\}, \min, +)$.

If $\mathcal{D}$ is a dioid, the set $\mathcal{D}^{n\times n}$ of $n \times n$ matrices with coefficients in $\mathcal{D}$ is also a dioid. The identity element of $\mathcal{D}^{n\times n}$ is also denoted e for any n. Note that n-dimensional row or column vector problems can be handled by embedding such vectors in square matrices with $n-1$ additional arbitrary (say, identically 'zero', i.e. ε) rows or columns.

Dioids can be endowed with a natural order: $a \geq b$ if $a = a \oplus b$. Then they become sup-semilattices and $a \oplus b$ is the least upper bound of a and b. Note that the natural order in $\mathbb{Z}_{\min}$ is just reversed with respect to the usual order. A dioid is *complete* if sums of infinite numbers of terms are always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all the elements of the dioid is denoted $\top$ (for 'top').

By adding $\top = +\infty$ (resp. $\top = -\infty$) to $\mathbb{Z}_{\max}$ (resp. $\mathbb{Z}_{\min}$), one obtains a complete dioid denoted $\overline{\mathbb{Z}}_{\max}$ (resp. $\overline{\mathbb{Z}}_{\min}$) ([3]).

A complete dioid (sup-semilattice) becomes a lattice by constructing the greatest lower bound of a and b, denoted $a \wedge b$, as the least upper bound of the (nonempty) subset of all elements which are less than a and b.

[2] The ratio of scalar variables has been discussed, but this ratio can be embedded in a more global 'division' of vector or matrix objects.

[3] Translate the equality $\varepsilon \otimes \top = \varepsilon$ into conventional notation in both $\overline{\mathbb{Z}}_{\max}$ and $\overline{\mathbb{Z}}_{\min}$.

3 FORWARD EQUATIONS

Event graphs is a special class of Petri nets for which places have only one transition upstream and only one downstream. Such graphs express synchronization in the form of forks and joins at transitions. Without loss of generality, firing times may be assumed to be zero and only places bear holding times. In the *dater* point of view, where $u_i(k)$ denotes the epoch or *date* at which the transition named u_i incurs its firing $\sharp k$, linear equations can be obtained, possibly after some manipulations, under the standard form

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k) \ , \\ y(k) &= Cx(k) \oplus Du(k) \ , \end{aligned} \tag{1}$$

where u is the vector of daters for source transitions, y is for sink transitions, x concerns internal transitions, and all calculations (in particular matrix/vector products) are to be understood in the dioid $\overline{\mathbb{Z}}_{\max}$. Similar equations can be obtained in the dioid $\overline{\mathbb{Z}}_{\min}$ using the *counter* point of view and variables $\overline{u}(t), \overline{x}(t), \overline{y}(t)$: for example, $\overline{u}_i(t) = \sup\{k \mid u_i(k) \leq t\}$ ([4]). Generally, the overline for counter symbols will be dropped and the same name u_i will be kept for the dater and the counter variables related to the same transition also named u_i: the only way to understand the context is to look at the argument which is either the event numbering k (thus daters are defined on the *event domain*) or the running time t (thus counters live in the more usual *time domain*). Note also that, in order to obtain equations in standard form for both daters and counters, it is generally necessary to use a state vector with different dimensionality in each point of view. The basic rule is the following: consider a transition x_i and assume that x_i is reached by a single arc and that this arc comes from transition x_j through a place with holding time α_{ij} and initial marking with β_{ij} tokens. Then equations in both points of view are

$$\begin{aligned} x_i(k) &= \alpha_{ij} \otimes x_j(k-\beta_{ij}) \ , \\ x_i(t) &= \beta_{ij} \otimes x_j(t-\alpha_{ij}) \ . \end{aligned} \tag{2}$$

Using the analogue of the z-transform for daters, based on the 'backward shift operator' γ in the event domain (formally, $\gamma u(k) = u(k-1)$), an input-output (transfer matrix) representation is obtained in the dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ of formal power series in γ with positive and negative exponents and coefficients in $\overline{\mathbb{Z}}_{\max}$. Similarly, another representation in $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$ is obtained from counters, based on the 'backward shift operator' δ in the time domain (formally, $\delta u(t) = u(t-1)$). Finally, a two-dimensional domain representation manipulates power series in both γ and δ with Boolean coefficients, with the conventional sum and product of power series, plus additional simplification rules (due to the fact that only non-decreasing trajectories of daters and counters are of interest):

$$\gamma^k \oplus \gamma^l = \gamma^{\min(k,l)} \quad \text{and} \quad \delta^t \oplus \delta^s = \delta^{\max(t,s)} \ .$$

This structure is denoted $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. In $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, Eq. (2) yields

$$X_i(\gamma,\delta) = \gamma^{\beta_{ij}}\delta^{\alpha_{ij}} X_j(\gamma,\delta) \ .$$

More generally, a timed event graph yields equations in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ of the form

$$X = \mathcal{A}X \oplus \mathcal{B}U \ , \quad Y = \mathcal{C}X \oplus \mathcal{D}U \ , \tag{3}$$

[4] Another seemingly less natural, but mathematically more convenient, definition is $\overline{u}_i(t) = \inf\{k \mid u_i(k) \geq t\}$. However, equations remain the same for this alternative definition.

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, X, U, Y$ are matrices or vectors with entries in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$, but those of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are *polynomial*. Since the *least* solution of the implicit equation $X = \mathcal{A}X \oplus b$ ('least' in the natural order of the dioid) is $X = \mathcal{A}^* b$ with $\mathcal{A}^* = e \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \dots$, one then obtains $Y = \mathcal{H}U$ with $\mathcal{H} = \mathcal{C}\mathcal{A}^*\mathcal{B} \oplus \mathcal{D}$: $\mathcal{H}$ is the *transfer matrix*.

The following observation will allow us to handle various representations and contexts with formally the same equations and calculations. Indeed, Eq. (1) are amenable to the form (3) by appealing to the γ-transform and by correctly interpreting the symbols $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, X, U, Y$. Similarly, this can also be viewed as a δ-transform representation. Consider now the situation of Eq. (1) with matrices A, B, C, D depending on k, which occurs for example along any sample path of a *stochastic* timed event graph. Suppose that k is confined to a finite interval $\{0, \dots, K\}$ ([5]). Then, it suffices to arrange the vectors, say $x(k), k = 0, \dots, K$, into a big vector X, and to similarly arrange the matrices, say $A(k), k = 0, \dots, K$, into appropriate big matrices, say $\mathcal{A}$, to arrive at the same formal equations (3).

4 RESIDUATION THEORY

A mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are ordered sets, is residuated if for all $y \in \mathcal{E}$, the least upper bound of the subset $\{x \mid f(x) \le y\}$ exists and belongs to this subset. It is then denoted $f^\sharp(y)$. The mapping $f^\sharp : \mathcal{E} \to \mathcal{D}$ is called the *residual* of f. When $\mathcal{D}$ and $\mathcal{E}$ are complete dioids, a mapping f is residuated if and only if $f(\varepsilon) = \varepsilon$ and f is lower-semicontinuous, that is,

$$f\left(\bigoplus_{i \in I} a_i\right) = \bigoplus_{i \in I} f(a_i) \qquad (4)$$

for any (finite or infinite) set I. A residual mapping is always upper-semicontinuous, that is, an equality such as (4) holds true for $f^\sharp$ with $\wedge$ replacing $\oplus$. Obviously, $f^\sharp(\top) = \top$. Any upper-semicontinuous mapping g satisfying this property is then *dually residuated* and its *dual residual* is denoted $g^\flat$: $g^\flat(x)$ is the greatest lower bound of $\{y \mid g(y) \ge x\}$ (and it belongs to this subset).

This theory can be applied to the mappings $x \mapsto a \otimes x$ and $x \mapsto x \otimes a$ in a complete (but not necessarily commutative) dioid. These mappings are residuated (but generally not dually residuated). The residual mappings will be denoted

$$y \mapsto a \backslash y = \frac{y}{a} \quad \text{and} \quad y \mapsto y / a = \frac{y}{a}$$

in one- and two-dimensional displayed expressions. When passing from a dioid $\mathcal{D}$ to the matrix dioid $\mathcal{D}^{n \times n}$, one has, for two matrices A and B, that

$$(A \backslash B)_{ij} = \bigwedge_k \frac{B_{kj}}{A_{ki}} \quad \text{and} \quad (B / A)_{ij} = \bigwedge_k \frac{B_{ik}}{A_{jk}} \ . \ (5)$$

The mnemonic way to remember these formulæ is to think of $A' \odot B$ and $B \odot A'$ respectively, where the prime denotes transposition and $\odot$ would be a special matrix product in which $\oplus$ is replaced by $\wedge$ and $\otimes$ is replaced by $\backslash$ and $/$ respectively. These formulæ extend without difficulty to nonsquare matrices (remember the trick to embed nonsquare into larger square matrices). In particular, for two n-dimensional column vectors X and Y, $X \backslash Y$ is a scalar whereas Y / X is a square matrix. This section is concluded by recalling some formulæ (see Table 1) which will be useful later on.

[5] Consider that $x(-1)$ is identically 'zero'—i.e. ε—so that $A(0)$ is irrelevant, say $A(0) = A(1)$ and $x(0) = B(0)u(0)$, that is, $u(0)$ is used to set the initial condition.

TABLE 1 Formulæ involving division

$$\frac{x \wedge y}{a} = \frac{x}{a} \wedge \frac{y}{a} \ , \qquad \frac{x \wedge y}{a} = \frac{x}{a} \wedge \frac{y}{a} \ , \qquad \text{(f.1)}$$

$$\frac{x}{a \oplus b} = \frac{x}{a} \wedge \frac{x}{b} \ , \qquad \frac{x}{a \oplus b} = \frac{x}{a} \wedge \frac{x}{b} \ , \qquad \text{(f.2)}$$

$$a\frac{x}{a} \le x \ , \qquad \frac{x}{a}a \le x \ , \qquad \text{(f.3)}$$

$$\frac{x}{ab} = \frac{a \backslash x}{b} \ , \qquad \frac{x}{ba} = \frac{x / a}{b} \ , \qquad \text{(f.4)}$$

$$\frac{a \backslash x}{b} = \frac{x / b}{a} \ , \qquad \frac{x / a}{b} = \frac{b \backslash x}{a} \ , \qquad \text{(f.5)}$$

If the underlying dioid in $\overline{\mathbb{Z}}_{\max}$, $a \backslash b = b / a = b - a$ (conventional minus sign) at least for finite a and b, and of course $\wedge$ is $\min$. However it should be noticed that $\varepsilon / \varepsilon = \top$, hence $-\infty - (-\infty) = +\infty$, whereas $\varepsilon \otimes \top = \varepsilon$, hence $-\infty + \infty = -\infty$. This shows the ambiguity of the conventional notation in some circumstances, and the dioid notation is recommended even for $\overline{\mathbb{Z}}_{\max}$.

5 BACKWARD EQUATIONS

The following problem, which will be formulated in the dater point of view, is addressed. A similar problem can be raised in the counter point of view, and in fact the two-dimensional domain approach can handle both view points at the same time. Given a sequence $\{z(k)\}_{k=0,\dots,K}$ of desired outputs, find the *latest* input sequence $\{u(k)\}_{k=0,\dots,K}$ such that the desired output is matched as closely as possible, that is, in more mathematical terms, given Z (the desired output sequence), find the maximum U (input sequence) such that $\mathcal{H}U \le Z$, where $\mathcal{H}$ is the transfer matrix of the system. The desired output sequence given in the interval $\{0, \dots, K\}$ can be completed for $k < 0$ by ε and for $k > K$ by $\top$ in order to consider its γ-transform Z. As already discussed, $u(0)$ indeed sets the initial condition in (1) as long as $x(k) = \varepsilon$ for $k < \varepsilon$. This may require using a special matrix B for $k = 0$ and thus one is led to consider again k-dependent matrices, a case that was discussed earlier. It was shown that whatever point of view is adopted, the system model can be described by equations of the form (3) with different interpretations of the underlying dioid and of the symbols $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, X, U, Y$.

Since the transfer mapping $U \mapsto \mathcal{H}U$ is residuated, the answer to the previous problem is $U = \mathcal{H} \backslash Z$. Let us make this formula more explicit. Since $\mathcal{H} = \mathcal{C}\mathcal{A}^*\mathcal{B} \oplus \mathcal{D}$, successive applications of (f.2) and (f.4) lead to

$$U = \frac{Z}{\mathcal{C}\mathcal{A}^*\mathcal{B}} \wedge \frac{Z}{\mathcal{D}} = \frac{\mathcal{A}^* \backslash (\mathcal{C} \backslash Z)}{\mathcal{B}} \wedge \frac{Z}{\mathcal{D}} \ .$$

Let $\Xi = \mathcal{A}^* \backslash (\mathcal{C} \backslash Z)$. It can be proved that Ξ is the *greatest* solution of the implicit equation below, and an internal representation of the mapping $Z \mapsto \mathcal{H} \backslash Z$ is obtained, namely,

$$\Xi = \frac{\Xi}{\mathcal{A}} \wedge \frac{Z}{\mathcal{C}} \ , \quad U = \frac{\Xi}{\mathcal{B}} \wedge \frac{Z}{\mathcal{D}} \ . \qquad (6)$$

Of course, by construction $Y = \mathcal{H}U \leq Z$. Remember that, in the counter interpretation (δ-transforms), this means that U produces *at least as much events* (firings at the output transitions) as required by Z. Returning to the dater interpretation over a finite interval $\{0, \ldots, K\}$, from (6) one derives the following recursive equations (matrices are assumed to be k-independent to alleviate notation):

$$\xi(k) = \frac{\xi(k+1)}{A} \wedge \frac{z(k)}{C} \quad , \quad u(k) = \frac{\xi(k)}{B} \wedge \frac{z(k)}{D} \ .$$

This form exhibits the backward nature of the recursion which starts at K with $\xi(K+1) = \top$. Moreover, because 'division' implies transposition of the matrix in the denominator (see (5)), the event graph is also swept from output to input transitions. These equations can be derived by direct reasoning on the graph, and written with standard notation $\min$ and $-$, with, however, the caution required to handle possible infinite terms. The quantity $\xi_i(k)$ indicates the latest date at which the internal transition x_i should incur its firing $\sharp k$ in order not to delay future outputs beyond the deadline provided by Z. It will be proved indeed that X, caused by that U specified by (6), is less than Ξ. Observe first that, because $(\mathcal{A}^*)^2 = \mathcal{A}^*$ and thanks to (f.4), one has that

$$\Xi = \frac{Z}{\mathcal{C}\mathcal{A}^*} = \frac{Z}{\mathcal{C}\mathcal{A}^*\mathcal{A}^*} = \frac{\mathcal{C}\mathcal{A}^* \backslash Z}{\mathcal{A}^*} = \frac{\Xi}{\mathcal{A}^*} \ .$$

Then, since $U \leq \mathcal{B} \backslash \Xi$,

$$X \leq \mathcal{A}^*\mathcal{B}\left(\frac{\Xi}{\mathcal{B}}\right) = \mathcal{A}^*\mathcal{B}\left(\frac{\frac{\Xi}{\mathcal{A}^*}}{\mathcal{B}}\right) = \mathcal{A}^*\mathcal{B}\left(\frac{\Xi}{\mathcal{A}^*\mathcal{B}}\right) \leq \Xi \ ,$$

where (f.4) was again used, the last inequality being due to (f.3).

6 MARGINS

A (nonnegative) difference $\xi_i(k) - x_i(k)$ represents the 'spare time' or the 'margin' which is available at transition x_i for the firing $\sharp k$, i.e. an exogenous event may delay this event by this amount without preventing the future deadlines to be met. The same quantity, in terms of counters (denoted $\xi_i(t) - x_i(t)$ according to our notational convention) is nonpositive (remember the reversed order in $\overline{\mathbb{Z}}_{\min}$) and it represents the number of firings which may 'accidentally be lost' up to time t. The translation of this information in terms of 'numbers of tokens which could be lost in the places' would require a more thorough study combining the results of (Max Plus, 1991) with the present results (both using differences, that is, 'divisions', of variables attached either to the same transition or to a pair of transitions located upstream and downstream the same place). This is beyond the scope of this short paper.

By returning to dioid notation, differences such as $\xi(k) - x(k)$ emerge from the calculation of $\Sigma = \Xi / X$ (whereas $\sigma = X \backslash \Xi$, as a scalar, carries less information). If e.g. X is interpreted as the concatenation of the whole trajectory $\{x(k)\}_{k=0,\ldots,K}$ into a single big column vector, then Σ is a big square matrix of dimension $n(K+1)$ (n being the dimension of $x(k)$) for which diagonal terms are of direct interest for us. Let us now establish an equation for Σ. One has that $X = \mathcal{A}^*\mathcal{B}U$ and $\Xi = \mathcal{C}\mathcal{A}^* \backslash Z$. Observe that (f.5) reads $(a \backslash x) / b = a \backslash (x / b)$: this is a kind of associativity property which makes parentheses useless. Using this and (f.4) repeatedly, one obtains

$$\Sigma = \frac{\mathcal{C}\mathcal{A}^* \backslash Z}{\mathcal{A}^*\mathcal{B}U} = \frac{\mathcal{C}\mathcal{A}^* \backslash (Z / U)}{\mathcal{A}^*\mathcal{B}} = \mathcal{A}^* \backslash \left(\mathcal{C} \backslash (Z / U) / \mathcal{B}\right) / \mathcal{A}^* .$$

Let $Q = Z / U = Z / (\mathcal{H} \backslash Z)$ (in the context of daters, this matrix contains information about the minimum time spent within the system since it involves differences between the target output and the latest input which matches this output). Then, it can be shown that Σ is the greatest solution of the following equation (which is reminiscent of a Lyapounov equation):

$$\Sigma = \mathcal{A} \backslash \Sigma \wedge \Sigma / \mathcal{A} \wedge \mathcal{C} \backslash Q / \mathcal{B} \ .$$

7 CONCLUSION

Finally, what has been considered here is a kind of optimal tracking problem (given the output trajectory $z(\cdot)$ to be tracked) and $x(\cdot)$, resp. $\xi(\cdot)$, plays the role of the state, resp. the co-state, vector of this 'optimal control problem'. The fact that, in a conventional LQ optimal control problem, the co-state is 'proportional' to the state via a Riccati matrix $P(\cdot)$, suggested us that the matrix $S(k) = \xi(k) / x(k)$ (diagonal block of Σ) might play the role of a Riccati matrix and a backward recursive equation satisfied by $S(\cdot)$ has been searched for. For the time being, this question remains open and it is unlikely that it can be answered positively without additional features (i.e. for any $z(\cdot)$).

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